LECTURE 4-3

We now prove Hensel's Lemma: *given a complete ring R with respect to an ideal I and $f \in R[x]$ such that $f(a) \cong 0 \mod f'(a)^2 I$, there is $b \in R$ with $f(b) = 0, b \cong a \mod f"(a)I$ and this b is uniquely determined if $f'(a)$ is a non-zero-divisor in R.* Set $f'(a) = e$. then there is $h(x) \in R[x]$ such that $f(a+ex) = f(a)+f'(a)ex+h(x)(ex)^2 = f(a)+e^2(x+x^2h(x))$. There is then a ring homomorphism ϕ from $R[[x]]$ to itself that is the identity on R and sends x to $x + x^2h(x)$, which is in fact an isomorphism, since the coefficient 1 of x is a unit. Applying ϕ^{-1} we get $f(a + e\phi^{-1}(x)) = f(a) + e^2x$. By hypothesis we have $f(a) = e^2c$ for some $c \in I$. Then there is an algebra homomorphism ψ from $R[[x]]$ to R that is the identity on R and sends x to $-c$. Applying it, we get $f(a + e\psi\phi^{-1}(x)) = 0$, so $b = a + e\psi(\phi^{-1}(x))$ is our desired element. Next suppose that e is a non-zero-=divisor and b, b_1 are roots of f differing from a by elements of eI, say $b = a + er, b_1 = a + er_1$ with $r, r_1 \in I$. Then there are ring homomorphisms β, β_1 from $R[[x]]$ to R that are the identity on R and send x to r, r_1, respectively. Applying them to the above formulas we get $0 = f(a)+e^2(r+r^2h(r)) = f(a)+e^2(r_1+r_1^2h(r_1))$. Subtracting and using the assumption that e is a non-zero-divisor we get $r + r^2h(r) = r_1 + r_1^2h(r_1)$, whence $\beta\phi(x) = \beta_1(\phi(x)$. By the uniqueness of β, β_1, we get $\beta\phi = \beta_1\phi$ and then $\beta = \beta_1$, since ϕ is an isomorphism. Hence $r = r_1$, as desired.

You will prove the classical version of Hensel's Lemma in upcoming homework: *let R be a complete with respect to an ideal I and let $F(x) \in R[x], f(x)$ its reduction mod I. If $f(x)$ factors in $R/I[x]$ as g_1g_2 with g_1 monic and g_1, g_2 generating the unit ideal, then there is a unique factorization $F(x) = G_1G_2$ in $R[x]$ G_1 monic and G_1, G_2 reducing to g_1, g_2, respectively, in $R/i[x]$.*

We now briefly return to algebraic geometry, treating one more topic in the first chapter of Hartshorne. Let V, W be affine varieties with coordinate rings R, S and let P, Q be points of V, W, respectively, corresponding to maximal ideals M, N of R, S. We say that P, Q are *analytically isomorphic* if the localized rings R_M, S_N become isomorphic when completed at M, N, respectively. As an example, let V be the nodal cubic curve in K^2 (K algebraically closed) with equation $y^2 = x^2(x + 1)$, let P be the origin, and let W be the reducible variety with equation $xy = 0$ wit Q again the origin. Then the varieties V and W are of course far from isomorphic and the points P, Q look rather different in these varieties, but nevertheless P and Q are analytically isomorphic. To prove this we must show that the quotient $K[[x, y]]/(y^2 - x^2 - x^3)$ is isomorphic to the quotient $K[[x, y]]/(xy)$. The key point is that the leading term $y^2 - x^2$ of $y^2 - x^2 - x^3$ factors as $(y - x)(y + x)$ and the two factors $y - x, y + x$ are linearly independent. Now we claim that there are power series g, h with $g = y + x + g_2 + g_3 + \ldots, h = y - x + h_2 + h_3 + \ldots$ with the g_i, h_i homogeneous of degree i, such that $y^2 - x^2 - x^3 = gh$. We construct them (as usual) step by step. To determine g_2, h_2 we must have $(y - x)g_2 + (y + x)h_2 = -x^3$, which is possible since $y - x, y + x$ generate the unique maximal ideal of $K[[x, y]]$; similarly we can construct g_3, h_3, and so on. Hence the completed localization $\hat{V}_M$ is isomorphic to $K[[x, y]]/(gh)$. but now g, h begin with linearly independent linear terms, so by a small extension of a previous result on isomorphisms from a power series ring to itself there is an automorphism of $K[[x, y]]$ sending g, h to x, y, respectively. Hence $K[[x, y]]/(y^2 - x^2 - x^3) \cong K[[xy]]/(xy)$, as desired; analytically, the singularity of V at P looks like that of two lines crossing; note also that the coordinate ring R and its localization R_M are integral domains, but the completion $\hat{R}_M$ is not. More generally, given any curve X in K^2 defined by the equation $f(x, y) = 0$ containing $(0, 0)$, so that f has zero constant term, write f as $f_r + f_{r+1} + \dots$ with the f_i homoogeneous of degree i and $f_r \neq 0$. By algebraic closure of K, f_r factors as the product of r linear terms; if none of these is a multiple of another, then we say that $(0, 0$ is an *ordinary r-fold point* of X. Then any two ordinary double points of any two curves are analytically isomorphic, by an argument similar to the above, as (it turns out) are any two ordinary triple points of any two curves. But the same is *not* true for any two 4-fold points. To see why, recall that nonzero linear combinations of the variables x and y, up to nonzero scalar multiple, are in bijection to the elements of $\mathbb{P}^1$, projective 1-space over K. The automorphism group $PSL_2(K)$ acts on this $\mathbb{P}^1$ 3-transitively, i.e. in a such way that given any six elements $a_1, a_2, a_3, b_1, b_2, b_3$ of $\mathbb{P}^1$ with the a_i and b_i distinct, there is a unique automorphism of $\mathbb{P}_1$, arising from an element of $PGL_2(K)$ sending the a_i to the b_i, but precisely because this automorphism is unique, the image of any $a_4 \in \mathbb{P}^1$ under it, is uniquely determined. Thus there is a one-parameter family of orbits of homogeneous polynomials f_r in x, y of degree 4 under the action of $PGL_2(K)$, and accordingly a one-parameter family of ordinary 4-fold singular points of curves up to analytic isomorphism.