LECTURE 4-28

Continuing from where we left off last time, we now suppose that we have a ring extension $A \subset B$ in which A and B are both integral domains, Our next result states that *if $x \in B$ is integral over an ideal I of A and A is integrally closed, then x is algebraic over the quotient field K of A and the nonleading coefficients of its minimal (monic) polynomial p over K all lie in $\sqrt{I}$.* Indeed, it is clear that x is algebraic over K. Let L be a finite extension of K containing all the conjugates of x (the other roots of p). Each of these conjugates x_i is also integral over I and the coefficients of p are polynomials in the x_i, so they too are integral over I, whence they too lie in $\sqrt{I}$ by the closedness of A. Now we can show that the Going-Down property holds for A and B in this situation: *given $Q_1 \in Spec\ B$ with contraction $P_1 \in Spec\ A$ and $P_2 \in Spec\ A, P_2 \subset P_1$, there is $Q_2 \in Spec\ B, Q_2 \subset Q_1$ such that Q_2 contracts to P_2.* To prove this it is enough to show that $B_{Q_1}P_2 \cap A = P_2$. If $x = y/s \in B_{Q_1}P_2$ then y is integral over P_2, whence its minimal polynomial over K, the quotient field of A, has all nonleading coefficients u_i in P_2. If in addition $x \in A$, then $s = yx^{-1}, x^{-1} \in K$, so the minimal polynomial of s over K has typical nonleading coefficient $v_i = u_i/x^i$, whence $x^i v_i = u_i \in P_2$. But s is integral over A, regarded as an ideal of itself, so each $v_i \in A$. If $x \notin P_2$, then each $v_i \in P_2$ and a suitable power of s lies in $BP_2 \subset BP_1 \subset Q_1$, forcing $s \in Q_1$, a contradiction. Hence $x \in P_2$, as desired.

A ring extension $A \subset B$ with B Noetherian has the going-up property if and only if the map Spec $B \to$ Spec A is closed (in the sense that it takes closed sets to closed sets), as you will show in homework. For general A and B, if this last map is open, then B has the going-down property over A. An example is given in Figure 10.4 in §10.2 of Eisenbud of a homomorphism not satisfying the going-down property.

We now turn to valuation rings, which generalize the discrete valuation rings we studied last quarter. Let A be an integral domain with quotient field K. We call A a *valuation ring* of K if for each $x \in K, x \neq 0$, either $x \in A$ or $x^{-1} \in A$ (or both). This condition says that A occupies a very large fraction of K. Note that it holds, for example, for the p-adic integers $\mathbb{Z}_p$, for given any power series $\sum_{k=-n}^{\infty} a_i p^i$ in $\mathbb{Q}_p$ but not $\mathbb{Z}_p$, we must have $n > 0$ (it must begin with a strictly negative power of p), whence its inverse begins with a positive power of p and lies in $\mathbb{Z}_p$. Any valuation ring A is local, for if M denotes the set of $x \in A$ with $x^{-1} \notin A$ or $x = 0$, then any $x \in M$ has $ax \in M$ for all $a \in A$, lest x^{-1}, as a multiple of $(ax)^{-1}$, lie in A; if $x, y \in M$ with x, y nonzero, then either $xy^{-1} \in A$ or $yx^{-1} \in A$, whence in both cases $x + y \in M$, since $x + y = (1 + xy^{-1})y = (1 + yx^{-1})x$. Also A is integrally closed, for if $x \in K$ is integral over A, so that $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ with the $a_i \in A$, then either $x \in A$ or $x = -(a_1 + a_2 x^{-1} + \ldots + a_n x^{1-n}) \in A$. We can construct valuation rings via Zorn's Lemma, as follows: given the field K, let Ω be an algebraically closed field. Let Σ consist of all pairs (B, f) where B is a subring of K and f a homomorphism from B into Ω. We partially order such pairs by inclusion. By Zorn's Lemma Σ has a maximal element, say (B, g). We claim that B is a valuation ring of K and $\ker g$ its maximal ideal. First note that $g(B)$ is a subring of a field and therefore an integral domain, so $M = \ker g$ is prime. We can extend g to a homomorphism from B_M

to Ω since $g(x) \neq 0$ for $x \notin M$; then maximality forces $B_M = B$, whence B is local with maximal ideal M.

Now let x be a nonzero element of K. Let $B[x]$ be the subring of K generated by B and x and $M[x]$ the extension of M in $B[x]$. Then *either* $M[x] \neq B[x]$ *or* $M[x^{-1}] \neq B[x^{-1}]$, for otherwise we would have equations $u_0 + \ldots + u_m x^m = v_0 + \ldots + v_n x^{-n} = 1$ for $u_i, v_i \in M$, in which we may suppose that the degrees m, n are as small as possible. Supposing for definiteness that $m \geq n$, multiply the second equation by x^n, to get $(1 - v_0)x^n = v_1 x^{n-1} + \ldots + v_n$. Since $v_0 \in M, 1 - v_0$ is a unit, and we may write x^n as a combination of lower nonnegative powers of x. Multiplying by x^{m-n} and plugging into the first equation, we get another equation with a lesser value of m, contradicting the way it was chosen.

We can finally show that B is indeed a valuation ring of K. Let $x \in K, x \neq 0$; we must show that $x \in B$ or $x^{-1} \in B$. By the above paragraph, we may assume that $M[x]$ is not the unit ideal of $B' = B[x]$, whence $M[x]$ is contained in a maximal ideal M', whose contraction in B must be M, so that we get an embedding $k = B/M \subset k' = B'/M'$. Since the image of x in k' generates it as a ring, k' must be finite algebraic over k. By the algebraic closure of Ω, the embedding $k \subset \Omega$ extends to one from k' into Ω, whence maximality of B forces $x \in B$, as desired. Thus K always admits at least one valuation ring, though for some K (e.g. finite fields) the only possibility for this ring is K itself. If we can choose Ω and a subring A of K with a non-injective map from A into Ω, however, then we can find a valuation ring of K different from K itself. More generally, given any integral domain A with a prime ideal $P \neq 0$, then it turns out that any subring A' of the quotient field K of A maximal subject to the condition that $PA' \neq A'$ turns out to be a valuation ring of K different from K.