LECTURE 4-26

We continue with Atiyah-Macdonald, now focussing on Chapter 5, which deals with integrality of one ring over another. Let A, B be rings with $A \subset B$. We have seen that there is a continuous map Spec $B \to$Spec A given by contraction (intersection with A). Thus we have the reverse of the situation we had in algebraic geometry last quarter; there we started with a morphism $V \to W$ of varieties and argued that it gave rise to an algebra homomorphism $K[W] \to K[V]$, which in fact determined the morphism uniquely. Now we cannot expect an arbitrary continuous map from Spec B to Spec A to arise from a ring homomorphism, or even to characterize the maps so arising, but we can hope to translate properties of B as an A-algebra to relations between their prime spectra.

We say that B has the *lying-over property* (with respect to A) if the map Spec $B \to$ Spec A is surjective; it has the *going-up property* if whenever $Q_1 \in$Spec B contracts to $P_1 \in$Spec A and $P_2 \in$Spec $A, P_2 \supset P_1$, then there is $Q_2 \in$Spec B containing Q_1 and contracting to P_2. It has the *going-down property* if given $Q_2 \in$Spec B with contraction $P_2 \in$Spec A and $P_1 \in$Spec $A, P_1 \subset P_2$, there is $Q_1 \subset Q_2, Q_1 \in$Spec B contracting to P_1. For example, if $A = \mathbb{Z}, B = \mathbb{Q}$, then B does not have the lying-over or going-up properties with respect to A, for the only prime ideal 0 of B contracts to 0 and no ideal of B contracts to any larger prime in A; but B does (trivially) have the going-down property with respect to A, simply because no prime ideal of A is properly contained in 0. If $A = \mathbb{Z}, B = \mathbb{Z}[i]$, then B has all three properties: recall that the prime ideals in B take the form (p) where p is a prime number congruent to 3 mod 4, together with the principal ideals $(a + bi)$ such that $a^2 + b^2 = p$ is prime in $\mathbb{Z}$ and congruent to 1 or 2 mod 4. Here the map Spec $B \to$Spec A is two-to-one half the time and one-to-one the other half (by contrast, in algebraic geometry, a finite map $V \to W$ between affine varieties V, W is generically k-to-one for some constant k).

We saw last quarter that if B is integral over A (in the sense that every $x \in B$ satisfies an integral dependence, that is, a monic polynomial with coefficients in A), then B satisfies the lying-over property; indeed, by passing to a quotient of A it is enough to show that some element of Spec B contracts to 0, but this follows immediately from a lemma proved last time, since $0^{ec} = 0$. In fact, we have a stronger property: *there are no inclusions among prime ideals Q in B contracting to a fixed prime P of A*; this follows by localizing, since an easy argument from last quarter shows that if B is integral over A, then B is a field if and only if A is. (Last quarter, in the algebro-geometric setting we had the extra property that B was finitely generated as an A-module, which further implies that only finitely many prime ideals in B contract to a fixed one in A.)

We saw last time that if B is flat over A then it satisfies the going-down property; if in addition it is faithfully flat then it satisfies the going-up and lying-over properties. We now exhibit a different condition which also implies the going-down property. For any ring extension $A \subset B$, the set of elements of B integral over A is a subring containing A called its *integral closure*. We say that A is *integrally closed* in B if it equals its own integral closure, and integrally closed (without qualification) if A is an integral domain and is integrally closed in its quotient field; of course in this case it may still admit proper integral extensions. For example, $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$, as we stated and used in the fall, for given a dependence $(r/s)^n + a_{n-1}(r/s)^{n-1} + \ldots + a_0 = 0$ with the a_i, r, s all integral and r/s in lowest terms, we can multiply this equation by s^n and deduce a contradiction if s has a prime factor. The same argument shows that *any unique factorization domain is integrally closed.* Returning to any ring extension $A \subset B$, let S be a multiplicatively closed subset of A. If $x \in B$ is integral over A and $s \in S$, then x/s is integral over A_S, as one sees by dividing a dependence for x by a suitable power s^n; conversely if b/s is integral over A_S, then multiplying a dependence by $(st)^n$ a suitable power of st, where t is the product of the denominators in the dependence, we get a dependence for bt over A. Hence *the integral closure of A_S in B_S is C_S, where C is the integral closure of A in B.* It follows easily that *a domain A is integrally closed if and only if A_P is integrally closed for all prime ideals P, or A_M is integrally closed for all maximal ideals M.*

We conclude by extending the notion of integrality and integral closure to ideals. Given an extension $A \subset B$ and an ideal I of A, we say that $x \in B$ is *integral over I* if it satisfies a dependence with all nonleading coefficients in I (as one would expect). The integral closure of I in B is the set of elements integral over I. We relate this to the integral closure of A by the following lemma: *let C be the integral closure of A in B and I and ideal of A. Then the integral closure of I in B is the radical $\sqrt{I^e}$ of the extension I^e of I in C.* To see this, let $x \in B$ be integral over I; then the dependence shows at once that $x^n \in I^e$ for some n, so that $x \in \sqrt{I^e}$. Conversely, if $x \in \sqrt{I^e}$ then $x^n = \sum a_i x_i$ for some $a_i \in I, x_i \in C$. Integrality of the x_i over A shows that the subring $M = A[x_1, \ldots, x_n]$ generated by A and the x_i is finitely generated as an A-module, and $x^n M \subset IM$. By our proof of the Cayley-Hamilton Theorem again, x^n is integral over I, whence so is x.