# LECTURE 4-19

Cotinuing with Chapter 6 of Eisenbud, we now investigate flatness in general. To do this we need to study tensor products in more detail. Let $M, N$ be $R$-modules with $N$ generated by a set $\{n_i\}$ of elements. We know that every element of $M \otimes N$ may be written as a finite sum $\sum_i m_i \otimes n_i$ with $m_i \in M$. Then we claim that *such a sum is 0 if and only if there are elements $m'_j$ in $M$ and elements $a_{ij} \in R$ such that $\sum_j a_{ij} m'_j = m_i$ for all $i$ and $\sum_i a_{ij} n_i = 0$ for all $j$.* Indeed, if elements $a_{ij}$ with these properties exist, then $\sum_i m_i \otimes n_i = \sum_i (\sum_j a_{ij} m'_j) \otimes n_i = \sum_j m'_j (\sum_i a_{ij} n_i) = 0$. For the converse, suppose first that $N$ is free and the $n_i$ form a basis; then the result is immediate since $M \otimes N$ is isomorphic to a direct sum of copies of $M$, one for each $n_i$, and the sum $\sum_i m_i \otimes n_i$ corresponds to the tuple $(m_1, , m_2 \ldots)$ in this isomorphism. In general, there are free modules $F, G$ over $R$, an exact sequence $F \to G \to N \to 0$, and elements $g_i \in G$ mapping to $n_i$ for all $i$. Then $\sum_i m_i \otimes g_i$ goes to 0 in this sequence, whence $\sum_i m_i \otimes g_i = \sum_j m'_j \otimes y_j$ with $y_j$ in the image of $F$ (so going to 0 in $G$). Writing each $y_j$ as a combination $\sum_i a_{ij} g_j$ of basis elements $g_j$ and applying the argument above to the difference $\sum_i m_i \otimes g_i = \sum_j a_{ij} \otimes g_i$ we deduce that $m_i = \sum_j a_{ij} m'_j$ while $y_j$ goes to $\sum_i a_{ij} n_i = 0$, as required. From this we get the *equational criterion for flatness: an $R$-module $M$ is flat if and only if for every relation $\sum_i n_i m_i = 0$ with $n_i \in R, m_i \in M$ there are elements $m'_j \in M$ and $a_{ij} \in R$ with $\sum_j a_{ij} m'_j = m_i$ for all $i$ and $\sum_i a_{ij} n_i = 0$ for all $i$*; this follows because the image of $\sum_i n_i \otimes n_i$ in the tensor product $I \otimes M$ of an ideal $I$ of $R$ and $M$ is 0 under the multiplication map if and ony if $\sum_i n_i m_i = 0$.

We can express this last condition by a commutative diagram. *An $R$-module $M$ is flat if and only if for every map $\beta$ from a finitely generated free module $F$ to $M$ and for every submodule $K$ of $\ker \beta$ generated by one element, there is a free module $G$, a map $\gamma : F \to G$, and a map $\pi : G \to M$ such that $\pi\gamma = \beta$ and $K \subset \ker \gamma$. The same holds for a submodule $K$ generated by finitely many elements. Finally, a finitely presented module is flat if and only if it is a summand of a free module, or equivalently it is projective.* Indeed, an element $f$ in the kernel of a map from a free module $F$ to $M$ is a relation among the images $m_j$ of the basis elements of $F$; the elements $m'_j$ of the Equational Criterion for Flatness correspond to a map from another free module $G$ taking teh generators of $G$ to the $m'_j$. A matrix with entries $a_{ij}$ such that $\sum_j a_{ij} m'_j = 0$ corresponds to a map $\gamma$ making the digram commute. The condition that $\sum_i a_{ij} n_i = 0$ in $R$ for all $j$ then says that $\gamma(f) = 0$, as required. If the map $\gamma$ exists with kernel containing one element of $K$, then composing it with other maps $\gamma'$ from $G$ to itself killing other elements lying in the kernel of $\beta$ and thus also the kernel of the composition $\pi\gamma$, we arrive at a new $\gamma$ with the required properties. Finally, if $M$ is finitely presented and flat then there is a surjection $\beta : F \to M$ from a finitely generated free module $F$ to $M$ whose kernel $K$ is finitely generated. Letting $\gamma, \pi$ be as above, so that $K$ lies in the kernel of $\gamma$, we find that the image of $\gamma$ is sent isomorphically to $M$ by $\pi$, so that $\pi$ splits and $M$ is a summand of $G$, as desired.

As another corollary, *let $k$ be a field, $R = k[t]$ the polynomial ring in one variable over $k$, and let $S$ be a Noetherian ring flat over $R$. If the fiber $S/tS$ over $(t)$ is a domain, and $U$ is the set of elements of the form $1 - ts$ for $s \in S$, then the localization $S_U$ of $S$ by $U$ is a domain.* To prove this we may replace $S$ by $S_U$ and assume at the outset and assume that all elements $1 - ts$ are already units in $S$. Let $I, J$ be ideals of $S$ with $IJ = 0$; we must show that $I$ or $J$ is 0. Enlarging $I$ and $J$ as necessary, we may assume that each is the annihilator of the other. Since $IJ \equiv 0 \mod (t)$ and $S/(t)$ is a domain we may assume that $J \subset (t)$. Then $J = J't$, where $J' = (J : t) = \{x \in S : xt \in J\}$. Since $t$ is not a zero divisor in $R$ and $IJ't = 0$, we know by the last lecture that $IJ' = 0$, whence $J' \subset J$ and $J = Jt$. By a corollary to the Cayley-Hamilton Theorem, we get $s \in S$ with $(1 - st)J = 0$, so $J = 0$, as desired.

Unfortunately one cannot avoid localization completely in the setting of the above paragraph; if $R = k[t], S = k[x, t] \times k[x, x^{-1}]$. The fiber over the maximal ideal $(t - a)$ for $a \in k$ is $S/(t - a)S$, which is a domain for $a = 0$ since $tk[t, t^{-1}] = k[t, t^{-1}$, but not for $a \neq 0$, and $S$ is not a domain either. We can avoid difficulties of this sort by working with graded rings.