LECTURE 4-17

Following Chapter 6 of Eisenbud, we now study families of algebras depending on a parameter and ask under what conditions they behave "nicely" (roughly meaning uniformly) with respect to this parameter. More precise, let R and S be rings with S and R-algebra, so that we have a ring homomorphism $R \to S$. For M a maximal ideal in R we define the *fiber over* M to be the R/M-algebra S/MS; more generally, for P a prime ideal in R we define the fiber $S(P)$ of S over P to be the algebra $K \otimes_R S$, where K is the quotient field of R/P. We want to study the dependence of $K \otimes_R S$ on P. We begin by looking at some simple examples; in all of them we take R to be $k[t]$, the polynomial ring in one variable over an algebraically closed field k. Obviously the nicest situation occurs when $R = S$; in this case all fibers $S(P)$ for P maximal are isomorphic to k, while the fiber $S(0)$ is the rational function field in one variable over k. While $S(0)$ is clearly infinite-dimensional over k and thus much bigger in one sense than the $S(P)$ for P nonzero, the dimension of any fiber $S(P)$ as a ring is 0, so for our purposes we regard the fibers as uniform. Next we take S to be $R[x]/(x^2 - t)$. In this case the fiber over $(t - a)$ is $k[x]/(x^2 - a)$, which is isomorphic to $k \oplus k$ if $a \neq 0$ and to $k[x]/(x^2)$ if $a \neq 0$. The fiber over (0) is $k(t)[x]/(x^2 - t)$, an extension of degree 2 of the residue field $k(t)$. Thus in all cases the fibers have degree 2 over the residue field; this is not surprising as S itself is free over R of rank 2. By contrast, take S to be $R[x]/(tx - t)$. Here the fibers vary wildly: if the prime P does not contain t, then t is a unit in K and $S(P) = K$, but if $P = (t)$, then $S(P) = k[x]$, so now the fibers do not all have the same dimension.

The key property of S that is present in the first two examples but not the last one is flatness. Let R be any ring. Recall that an R-algebra, or more generally an R-module M, is flat if and only if tensoring with M is an exact functor from R-modules to R-modules. Since the only possible obstruction to exactness occurs at the left end of a short exact sequence, it is equivalent to require that the induced map $M \otimes_R N' \to M \otimes_R N$ is an injection whenever we have an injection $N' \to N$ of R-modules. In fact, as we saw in the fall, it is enough to require that the multiplication map $I \otimes_R M \to M$ be an injection for every finitely generated ideal I of R. We also learned in the fall that projective modules are flat. The definition of localization for rings and modules shows that any localization of R is flat as an R-module.

There is a precise way to measure how far a general R-module is from being flat, or equivalently the failure of exactness of tensoring with the module. Given R-modules M, N, we define their *Tor groups* $\mathrm{Tor}_i^R(M, N)$ by starting with a projective (in particular a free) resolution $P_i \to \ldots \to P_0 \to M$, tensoring the P_i with N, and then taking homology, so that $\mathrm{Tor}_i^R(M, N)$ is the kernel of the map from $P_i \otimes N$ to $P_{i-1} \otimes N$ (taking P_{-1} to be 0) modulo the image of the map from $P_{i+1} \otimes N$ to $P_i \otimes N$. (If R is noncommutative, as we allowed it to be in the fall, then we lose the R-module structure on the Tor groups, which are just abelian groups, but if as here R is commutative, then the Tor groups retain the R-module structure. These groups are analogous to the Ext groups we defined in the fall.) In particular, if $x \in R$ is a non-zero-divisor and M a free R-module, then an obvious free resolution of the quotient $R' = R/(x)$ shows that $\mathrm{Tor}_0(R', M) = M/xM$, $\mathrm{Tor}_1(R', M) = \{m \in M : xm = 0\}$, while $\mathrm{Tor}_i(R', M) = 0$ for $i \geq 2$. Thus $\mathrm{Tor}_0(M, N)$ (we omit the ring R from the notation if it is understood from context) is just $M \otimes N$

itself while the other groups $\mathrm{Tor}_i(M,N)$ are to be regarded as higher derived functors of the tensor product. If R is Noetherian and M,N are finitely generated, then so are the Tor groups as R-modules. Given a short exact sequence $0 \to M' \to M \to M" \to 0$ of R-modules, we get a long exact sequence $\mathrm{Tor}_i(M',N) \to \mathrm{Tor}_i(M,N) \to \mathrm{Tor}_i(M",N) \to \dots$ $\mathrm{Tor}_1(M",N) \to M' \otimes N \to M \otimes N \to M" \otimes N \to 0$. The R-module M is flat if and only if $\mathrm{Tor}_1(R/I,M) = 0$ for every finitely generated ideal I of R.

Thus in particular if x is a non-zero-divisor in R and M is flat over R, then x must not be a zero divisor on M, for then the injection $R \to (x)$ given by multiplication by x must remain an injection upon tensoring with M This explains why the R-algebra S in our third example is not flat over R, as $t \in R$ becomes a zero divisor in S; by contrast, if we set $S = R[x]/(tx-1)$, then S is a localization of R and so flat over it. If R is a PID, then the above computation of $\mathrm{Tor}1(R/(x),M)$ shows that M is flat if and only if M is torsion-free.