LECTURE 4-14

Following §15.5 of Dummit and Foote or Exercises 15ff. of Chapter 1 of Atiyah-Macdonald, we define an analogue of the Zariski topology (in fact usually called exactly that) for a general commutative ring R. The *(prime) spectrum* of R, denoted Spec R, is by definition the set of its prime ideals. We topologize it much as we did K^n for K an algebraically closed field, by decreeing that the closed sets $V(S)$ are exactly those primes containing a fixed subset S of R. As with the Zariski topology on K^n, one easily checks that finite unions and arbitrary intersections of closed sets are closed, so Spec R is indeed a topological space. Again as before, the closed set $V(S)$ coincides with $V(I), I$ the ideal generated by S, and also with $V(\sqrt{I})$, so in the end it suffices to look at prime ideals containing a fixed radical ideal. We have seen that the intersection of all prime ideals containing any ideal I is the radical of I, so we immediately get the analogue of the Nullstellensatz in this setting, stating that the map $I \to V(I)$ defines an order-reversing bijection between radical ideals in R and closed subsets of Spec R. Now from the example of K^n we are used to non-Hausdorff topologies, but Spec R takes the failure of separation axioms to a new extreme, for if R is an integral domain then 0 is a prime ideal lying in every other, so the closure of the point 0 in Spec R is all of Spec R! We call the 0 ideal (and analogues of it for other rings) a *generic point*.

Now for $R = K[x_1, \ldots, x_n]$ with K an algebraically closed field we find that Spec R contains the points of K^n (corresponding to the maximal ideals in R) and the topology on these points is the Zariski topology of K^n, but now we get a whole family of new points corresponding to the irreducible subvarieties V of K^n, the closure of any such point consisting of the points corresponding to subvarieties W of V. It is important however to broaden this example by considering arbitrary fields K. For $n = 1$ we have the generic point together with one closed point for every monic irreducible polynomial f in $K[x]$; similarly for $R = \mathbb{Z}$ we have the generic point plus one closed point for every prime $p > 0$.

To understand what happens for $n = 2$ (and $R = K[x, y]$) it is helpful to make a general observation. Given any ring homomorphism $f : R \to S$ one checks immediately that the inverse image $f^{-1}(Q)$ is prime in R whenever Q is prime in S (by contrast the direct image $f(P)$ for P prime in R need not be prime), so we get a map f^* from Spec S to Spec R which is easily seen to be continuous. In particular the intersection of any prime ideal of R with any subring S of R is prime in S, so that if we know the prime ideals in S then we can say a lot about those in R. Another powerful technique is localization: given any multiplicatively closed subset U of R, the prime ideals of R_U are in order-preserving 1-1 correspondence with the prime ideals of R not meeting U and may well be easier to understand than the full set Spec R. In particular, for $R = K[x, y]$ a prime ideal P in R will either intersect $S = K[x]$ trivially and thus correspond to a prime ideal in the polynomial ring $K(x)[y]$ in one variable over the rational function field $K(x)$, which must be principal, or else P meets S in $Q = (f), f$ monic irreducible in S, and then once again P corresponds to a prime ideal in a polynomial ring in one variable, this time over the field $K[x]/(f)$. The upshot (using Gauss's Lemma to understand how a principal ideal in $K(x)[y]$ intersects $K[x, y]$) is that the nonzero prime ideals of R are either principal, generated by an irreducible polynomial in two variables, or maximal (having finite codimension in $K[x, y]$). A very similar picture holds for $R = \mathbb{Z}[x]$: every nonzero prime ideal is generated by either a single prime integer

p, or a single irreducible polynomial f in R, or else by a prime integer p and a polynomial f reducing mod p to an irreducible polynomial $\bar{f}$ in $\mathbb{Z}_p[x]$. The topology of Spec R is determined by the inclusion relations among these ideals, which are easy to work out.

There are a number of general properties of Spec R which are quite entertaining to derive. For example, let's work out the condition for Spec R to be disconnected as a topological space. This happens if and only if there are ideals I, J such that every prime ideal contains exactly one of the ideals I, J (and not the same one for every prime). Then the sum $I+J$ must be the unit ideal, so there is $e \in I$ with $1-e \in J$. But then every prime ideal contains $e(1-e)$, forcing $e^n(1-e)^n = 0$ for some n, but neither e^n nor $(1-e)^n$ is 0. But then no prime ideal can contain both e^n and $(1-e)^n$, whence these two powers also generate the unit ideal, and any element of the intersection $(e^n) \cap (1-e)^n$ is annihilated by $(1-e)^n$ and e^n, hence by all of R and is 0. The Chinese Remainder Theorem then guarantees that R is the direct sum of its quotients $R/(e^n)$ and $R/(1-e)^n$. Hence finally there must be an idempotent element f of R different from 0 and 1 (so that $f^2 = f$. Conversely, it is not difficult to check that any ring R with such an idempotent has Spec R disconnected.