LECTURE 4-12

Following Chapter 8 of Atiyah-Macdonald, we now turn attention to an important but very special class of commutative rings, analogous to but even more special than the Dedekind domains we studied last quarter. Call ring R *Artinian* if it satisfies the *descending* chain condition on ideals, or equivalently if every nonempty set of ideal has a minimal element. If R is Artinian and $x \in R$ is not a zero divisor, then the descending chain $(x) \supset (x^2) \supset \dots$ must stabilize, say at (x^n), and then we must have $x^n = x^{n+1}y$ for some $y \in R$, forcing $xy = 1$, since x is not a zero divisor. Thus every non-zero-divisor is a unit and in particular *a prime ideal in an Artinian ring must be maximal*. Next look at the set of all finite intersections $\cap_i M_i$ of maximal ideals M_i; if $\cap_{i=1}^n M_i$ is a minimal ideal among these, then any maximal ideal M must contain this intersection (lest there be an even smaller one), forcing M to contain and thus equal M_i for some i. Thus *an Artinian ring has only finitely many maximal ideals, and thus only finitely many prime ideals*. Letting N be the nilradical (the intersection of the prime ideals), the descending chain $N \supset N^2 \supset \dots$ again stabilizes, say at $N^k = N^{k+1}$. If $N^{k+1} \neq 0$, then there is $x \in N$ with $N^k x$ minimal nonzero (among all nonzero $N^k z$ as z runs over N); since $N^{k+1}x = N^k x \neq 0$, there is $y \in N$ with $N^k yx \neq 0$, whence minimality forces $N^k yx = N^k x \neq 0$. Multiplying by a power of y on the left, we get $N^k y^n x = N^k x \neq 0$ for all n; but we must have $y^n = 0$ for some n since N (the radical of the 0 ideal) consists of nilpotent elements. This contradiction forces N to be nilpotent. Now if $M_1, \dots, M_n$ are the maximal ideals of our Artinian ring R, then we have $M_1^k \cdots M_n^k = 0$ for some k. The sum of any M_i^k and the product of the kth powers of the others does not lie in any maximal ideal, so must be the whole ring; applying the Chinese Remainder Theorem, we see that the intersection of the M_i^k, like their product, must be 0, and R is the direct sum of its quotients $R?M_i^k$, each of which is an Artinian local ring.

Thus any Artinian ring is a finite direct product of Artinian local rings R/M^k. Now the descending chain condition on ideals of R is inherited by its subquotients M^i/M^{i+1} for $0 \leq i \leq k-1$, whence each of these subquotients must be a finite-dimensional vector space over the field R/M. But this guarantees that ideals in each quotient also satisfy the ascending chain condition, so *any Artinian ring is also Noetherian*. Conversely, a Noetherian ring of dimension 0 (so that all prime ideals are maximal) is Artinian, for then the 0 ideal is a finite intersection of primary ideals, each of them having maximal radical and containing a power of that radical, whence we see as above that any such ring is a finite direct product of local rings with the same properties and we can imitate the above discussion to see that R is Artinian.

In particular if a radical ideal I of $K[x_1, \dots, x_n]$ is such that $K[[x_1, \dots, x_n]/I$ Artinian (for K algebraically closed) then I is the product of finitely many distinct maximal ideals, so that its associated variety consists of finitely many points; the quotient is the direct product of finitely may copies of K. Such quotients are of course not very interesting either algebraically or geometrically in and of themselves, but what is more interesting to observe is that Artinian rings can still be singular. Indeed, the easy examples of $\mathbb{Z}/(p^n)$ for p a prime and $K[x]/(x^n)$ (or its localization $K[[x]]/(x^n)$ are both singular. In fact, the only nonsingular Artinian local rings are fields (since the maximal ideal M must be such that $M/M^2 = 0, M = M^2 = 0$ by Nakayama's Lemma). The above examples arise as quotients of the discrete valuation rings $\mathbb{Z}_p$ (the p-adic integers, in this context) and $K[[x]]$ that we have seen earlier; both of these rings are also complete. If the maximal ideal M of an Artinian local ring R is principal, say generated by $x \in R$, then every nonmultiple of x in R is a unit, whence every element of R takes the form $x^n u$, where u is a unit. Such rings behave very much like discrete valuation rings, but with the crucial difference that the generator x of the maximal ideal is nilpotent. It is also possible for the maximal ideal in an Artinian local ring to require arbitrarily many generators, e.g. the quotient $K[x_1, \dots, x_n]/M^k$, where M is the augmentation ideal $(x_1, \dots, x_n)$ generated by the variables.

So powerful is the descending chain condition that it has many important consequences even for noncommutative rings. We call a noncommutative ring R Artinian if it satisfies the descending chain condition on left ideals; it turns out that it is equivalent to impose this condition on right ideals. The analogue of the Jacobson radical (the intersection of the maximal ideals) for such a ring (and indeed for any noncommutative ring) is again called the Jacobson radical and is the intersection of the annihilators Ann M of the irreducible right modules (which turns out to coincide with the intersection of the annihilators of the irreducible left modules). This radical is again nilpotent for Artinian rings, by an argument quite similar to the commutative case; if it is 0, we call the ring *semisimple*. A semisimple Artinian ring R s then a finite direct product of matrix rings over division rings; the main example, studied in the fall, is the group algebra of a finite group over a field K of characteristic 0. If R is a finite-dimensional K-algebra with K algebraically closed, then the division rings occurring in the decomposition are all isomorphic to K itself, as we saw earlier for complex group algebras.