LECTURE 4-10

We wrap up primary decomposition with a couple of further remarks and examples. First, in the situation where M is a finitely generated module over a Noetherian ring R, we know by definition that whenever Ann m is prime, for any $m \in M$, then Ann m is one of the associated primes of M; but in fact, for any $m \neq 0$, if Ann m is not prime, then by looking at an ideal maximal in the set of all Ann xm as x runs over the elements of R such that $xm \neq 0$, we get a prime ideal containing Ann m and lying in Ass M. It follows that *the union of the primes in Ass M coincides with the set of zero-divisors on M*. Moreover, if an ideal I consists solely of zero-divisors on M, then I lies in the union of the primes in Ass M and so by prime avoidance in just one of them. Thus *every ideal of R either contains a non-zero-divisor on M or annihilates a nonzero element of M*. A similar argument using finiteness shows that Ass M commutes with localization: *if S is any multiplicatively closed subset of R, then Ass M_S consists exactly of the prime ideals P_S such that $P \in$ Ass M and $P \cap S = \emptyset$.*

Next we look at the behavior of primary decomposition for graded modules over graded rings. the basic result states that *if $R = \oplus R_i$ is graded Noetherian and $M = \oplus M_i$ is a finitely generated graded R-module, then any ideal $P =$ Ann m for $m \in M$ that is prime is homogeneous and equal to the annihilator of a homogeneous element*. To prove this write any $f \in R$ as $\sum_{i=1}^{s} f_i$, where $f_i \in R_{d_i}$ and $d_1 < \cdots < d_s$. If $f \in P$ we will show that $f_1 m = 0$; by induction this will show that P is homogeneous. Write m as $\sum_{i=1}^{t} m_i$ with $m_i \in M_{e_i}$ and $e_1 < \cdots < e_t$. Argue by induction on t. First, fm is the sum of $f_1 m_1$ and terms of higher degree, so $f_1 m_1 = 0$. Then $f_1 m$ is a sum of fewer homogeneous terms than m, so by induction its annihilator I is homogeneous. If $P = I$, then we are done; otherwise choose $g \in I, g \notin P$; then $gf_1 \in P$ but $g \notin P$, so $f_1 \in P$, as desired. Now since P is homogeneous we have $Pm_i = 0$ for all i. But then $P \supset \cap_i \text{Ann } m_i, P = \cap_i \text{Ann } m_i \supset \prod_i \text{Ann } m_i$ and by primeness $P =$ Ann m_i for some i, as claimed. It follows that all associated primes of any such graded module are homogeneous, a primary decomposition of any graded submodule can be given using only graded submodules, and we can filter any such module M as $M_0 = 0 \subset \ldots \subset M_n = M$ in such way that the quotients $M_i/M_{i-1} \cong R/P_i$ have P_i homogeneous (as already observed last quarter).

Finally, we look at symbolic powers $P^{(n)}$ of prime ideals P, as defined in the last lecture, and say how they behave in algebraic geometry while also computing an explicit example where $P^{(2)}$ is strictly larger than P^2. If K is an algebraically closed field and P is a prime ideal in $K[x_1, \ldots, x_n]$ corresponding to a subvariety V of K^n, then it is known that *$P^{(n)}$ consists exactly of the elements f of the coordinate ring $K[V]$ vanishing to order at least n at every point of V, in the sense that $f \in M^n$ for the maximal ideal M of $K[V]$ corresponding to any point of V*. Thus $P^{(n)}$ indeed contains P^n but can be strictly larger. For a "naturally occurring" example where this happens, take x_{ij} for $1 \leq i, j \leq 3$ to be a set of independent variables and K an algebraically closed field. Form the *generic matrix* G whose ij-th entry is x_{ij}. The radical P of the ideal $I_2(G)$ generated by all the 2×2 minors of G is prime, as its corresponding variety V is the set of all 3×3 matrices over K of rank at most 1 and we can realize V as the closure of the image of the product $X = GL_3(K) \times GL_3(K)$ under the morphism f sending a pair (g, h) in this product to

gmh, where m is a fixed 3×3 matrix over K of rank 1, and $GL_3(K)$ is a principal Zariski-open subset of K^9 (defined by the nonvanishing of the determinant) and so is (isomorphic to) an irreducible affine variety. Thus V is irreducible, as a decomposition of it into proper closed subsets would decompose $X \times X$ into proper closed subsets (by applying f^{-1}) and $X \times X$ is irreducible. (In fact $I_2(G)$ is already prime, being equal to its radical, but we will not need this result.) Now we claim that $g = \det G$ lies in $P^{(2)}$; clearly it does not lie in P^2 as it is homogeneous of degree 3 but P^2 is spanned by homogeneous polynomials of degree at least 4. To show that $g \in P^{(2)}$, it suffices to show that $x_{11}g \in I_2(G)^2$, since clearly $x_{11} \notin \sqrt{I_2(G)}$. Multiplying the second and third columns of G by x_{11} and row-reducing, we see that $x_{11}^2 g$ is the product of x_{11} and a 2×2 determinant of matrix whose entries are 2×2 minors of G, so $x_{11}g$ lies in $I_2(G)^2$, as claimed. (Note that this result agrees with the characterization of symbolic nth powers of prime ideals in polynomial rings above, since the partial derivatives of $\det G$ with respect to each of its variables are 2×2 minors of G, so lie in $I_2(G)$). In fact it can be shown that $P^{(2)}$ is generated by P^2 and g in this case; there are more general formulas for the symbolic powers of prime ideals generated by minors of generic matrices of any fixed size.