FINAL EXAM SOLUTIONS–MATH 506A

1. Classify as completely as you can the fields that are homomorphic images of $\mathbb{Q}[x, y, z]$, the polynomial ring in three variables over $\mathbb{Q}$.

Any such field, being a finitely generated $\mathbb{Q}$-algebra, must be a finite extension of $\mathbb{Q}$ by Noether normalization; conversely, any such extension, being in fact generated a single element over $\mathbb{Q}$, is a homomorphic image of $\mathbb{Q}[x]$ or $\mathbb{Q}[x, y, z]$.

2. How many similarity classes are there of 4×4 matrices over $\mathbb{Q}$ whose minimal polynomial is $(x-1)^2$ and whose characteristic polynomial is $(x-1)^4$? Give a representative from each class.

Any such similarity class is determined by its Jordan form, which either has two 2×2 blocks or a single 2×2 block together with two 1×1 blocks (in each case with eigenvalue 1; thus there are two such classes.

3. Let p be a prime and let G be a group of order p^3 whose center Z has order p. Find the degrees (=dimensions) of all the irreducible representations of G, noting that G/Z is abelian.

Since G/Z is ableian and of order p^2, it has p^2 one-dimensional representations, each of which is also a one-dimensional representation of G. The remaining representations must have degree a power of p whose square is less than p^3, whence this degree must be p. As there must be at least one representation of degree larger than one (G being nonabelian) there must be exactly $p-1$ such representations, each of degree p.

4. Give a careful definition of the cohomology groups $H^n(G, A)$ of a finite group G with coefficients in a G-module A.

We have $H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, the nth Ext group of the trivial module $\mathbb{Z}$ over the integral group ring $R = \mathbb{Z}G$ with coefficients in the G-module A (computed via a projective resolution of $\mathbb{Z}$ as an R-module).

5. Let k be an algebraically closed field. Show that there is no injective k-algebra homomorphism from the polynomial ring $k[x_1, \ldots, x_n]$ to $k[x_1, \ldots, x_m]$ if $n > m$.

This was the hardest problem. If there were such a homomorphism, there would be n elements of $k[x_1, \ldots, x_m]$, or of its quotient field, algebraically independent over k, contradicting $\dim k^m = m$.

6. Show that the induced map Spec $\mathbb{Z}[i] \to$ Spec $\mathbb{Z}$ arising from the inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$ is surjective.

This follows by a result in class, since $R = \mathbb{Z}[i]$ is integral over $\mathbb{Z}$ (or else argue directly as in class that every nonzero prime ideal (p) in $\mathbb{Z}$ is either such that (p) remains prime in R (if $p \equiv 3 \mod 4$) or else is the contraction of the prime ideal $(a + bi)$ in R, where $a, b \in \mathbb{Z}$ are such that $a^2 + b^2 = p$).

7. Classify the finitely generated projective modules over $\mathbb{Z}$.

Any such module, being a submodule of a free $\mathbb{Z}$-module, must itself be free, and conversely any free $\mathbb{Z}$-module is projective over $\mathbb{Z}$.

8. Show that the polynomial ring $\mathbb{Z}[x]$ is not a Dedekind domain but $\mathbb{Q}[x]$ is.

$\mathbb{Z}[x]$is not a Dedekind domain since its Krull dimension is 2 (the nonzero prime ideal (2) is not maximal); $\mathbb{Q}[x]$ is a Dedekind domain since it is Noetherian with Krull dimension one and is integrally closed (being a UFD).