Lecture 5-31: Review, part 3

May 31, 2024

The final exam will take place on Monday, June 3, from 8:30 to 10:20 (note earlier start time) in the usual room. You should bring your own paper; you are allowed one page of handwritten notes on both sides. Only material in Dummit and Foote will be covered.

The Zariski topology is generalized to the setting of an arbitrary commutative ring R (in practice, usually Noetherian) as follows. The (prime) spectrum Spec R of R is the set of its prime ideals; this is the underlying topological space. The closed sets are then the set $\mathcal{Z}(I)$ of prime ideals containing a fixed ideal I (or equivalently a set of generators of I). The Nullstellensatz in this setting asserts the existence of inverse inclusion-reversing bijections between the sets of radical ideals of R and of nonempty closed subsets of Spec R; here a radical ideal I corresponds to the set $\mathcal{Z}(I)$ of prime ideal containing it while a closed subset C of Spec R corresponds to the intersection $\mathcal{I}(C)$ of the prime ideals in C.

Any closed subset of Spec R, like any closed subset of affine n-space $\mathbf{A}^n = k^n$, is then uniquely the union of finitely many irreducible closed subsets, none of them contained in any other; the irreducible closed subsets correspond under the Nullstellensatz to the prime ideals. The points of Spec R corresponding to maximal ideals M are then exactly the ones such that the singleton set $\{M\}$ is closed; they are not surprisingly called closed points. The point $\{0\}$, whose closure is all of Spec R, is called generic.

Thus if $R = P_n = k[x_1, \ldots, x_n]$, where k is an algebraically closed field, then the points of Spec R correspond to the subvarieties of $\mathbf{A}^n$, with the closure of the point corresponding to V containing the point corresponding to W if and only if $W \subset V$. In general one attempts to understand Spec R by relating it to Spec S for S a suitable subring of R, exploiting the continuous map from Spec R to Spec S given by contraction (sending a prime ideal P to $Q = P \cap S$). For example, if $R = \mathbb{Z}[i]$, the ring of Gaussian integers, then we take $S = \mathbb{Z}$. The nonzero ideals in Spec R are then of three types. If $p \in \mathbb{Z}$ is prime with $p \equiv 3 \mod 4$, then Rp is also prime in R; here the fiber over the principal ideal $(p) \subset S$ is a single point. If $p \equiv 1 \mod 4$, then Rp is the product of two distinct prime ideals $R(a + bi)$ and $R(a - bi)$, where $a, b \in \mathbb{Z}$ satisfy $a^2 + b^2 = p$; here the fiber over $(p) \subset S$ has two points. Finally, the exceptional prime $2 \in \mathbb{Z}$ is such that $2R = (R(1 + i))^2$, the square of a prime ideal. The fiber over $(2) \subset S$ again consists of just one point.

The ring $\mathbb{Z}[i]$ is a particularly nice example of a Dedekind domain, that is, an integrally closed integral domain of Krull dimension one. More generally, the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in any finite Galois extension K of $\mathbb{Q}$ is also a Dedekind domain. In a Dedekind domain any nonzero ideal I is a finite product $P_1 \cdots P_m$ of prime ideals, where this product is unique up to reordering the factors. What makes $\mathbb{Z}[i]$ (or more generally any PID) especially nice even among Dedekind domains is that it is a UFD; that is, any nonzero, nonunit element is a finite product of primes, unique up to multiplying the factors by units and reordering them. On the other hand, $\mathbb{Z}[x]$ is not quite a Dedekind domain, since it admits nonzero nonmaximal prime ideals (such as the one generated by 2). Nor, as you saw in homework, is $\mathbb{Z}[\sqrt{-7}]$ a Dedekind domain, since it fails to be integrally closed.

A very special (but very important) class of Dedekind domains) consists of the local ones, called DVRs (standing for discrete valuation rings). These are PIDs having a unique prime element x up to multiplication by units. In any such ring the only nonzero ideals are the principal ones (x^n) generated by powers of x, The two simplest examples of such rings are the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ by a nonzero prime ideal (p) and the power series ring $k[[x]]$ in one variable x over a field k.

Good luck and have a good summer!