Lecture 5-3: The cup product and Brauer group

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We wrap up the material on group cohomology with a brief account of its ring structure, given by the cup product, and give another setting in which factor sets arise.

As many of you know, groups are not the only mathematical objects with cohomology groups attached to them. Given any topological space X and a commutative ring R, one has homology groups $H_n(X, R)$ and cohomology groups $H^n(X, R)$ of X with coefficients in R. These have the structure of R-modules, not just abelian groups; in addition, the latter groups have a ring structure under something called the *cup product*. The sum $H^*(X, R) = \bigoplus_n H^n(X, R)$ thereby acquires the structure of a graded R-algebra, since the cup product of $H^n(X, R)$ and $H^m(X, R)$ lies in $H^{n+m}(X, R)$. In the setting of group cohomology, assume for simplicity that G acts trivially on a field k; then the group cohomology ring $H^*(G, k) = \bigoplus_n H^n(G, k)$ is a graded k-algebra. It is graded commutative in the sense that $xy = (-1)^{ij} yx$ if $x \in H^i(G, k), y \in H^j(G, k)$.

Take first $k = F_2$, the field of order 2; here graded commutativity is the same as commutativity. Also let $G = \mathbb{Z}_2$, the cyclic group of order 2. You have seen that $H^n(G, F_2) \cong \mathbb{Z}_2$ for all n; now I can say more precisely that the ring $H^*(G, k)$ is isomorphic to the polynomial ring $k[x]$ in one variable, where the degree of x is 1 (as usual). For odd primes p, the story is a bit more complicated; taking $G \cong \mathbb{Z}_p$ to be cyclic of order p and F_p to be the field of this order, we still have $H^n(G, k) \cong \mathbb{Z}_p$ for all n, but now it is the sum $\bigoplus_n H^{2n}(G, k)$ of the cohomology groups of even degree that is isomorphic to the polynomial ring $k[x]$, taking the degree of the variable x to be 2, while the sum $\bigoplus_n H^{2n+1}(G, k)$ can be realized as $k[x]y$, where the degree of y is 1 and $y^2 = 0$.

Returning to the case $k = F_2$, suppose now that G is replaced by the product $\mathbb{Z}_2^n$ of n copies of $\mathbb{Z}_2$. Then the *Künneth formula* in algebraic topology or group cohomology asserts that $H^*(G, k)$ is isomorphic to the tensor product $\otimes^n k[x]$ of n copies of $k[x]$, which may be identified with the polynomial ring $k[x_1, \ldots, x_n]$; here (again as usual) we take the degrees of all variables x_i to be 1. Thus if $n = 2$, we find that $H^2(G, k)$ has basis $x_1^2, x_1 x_2, x_2^2$ over k, so its cardinality is 8, as previously computed. A similar nth tensor power formula holds for $H^*(\mathbb{Z}_p, \times\mathbb{Z}_p, F_p)$.

Besides group extensions, central simple algebras provide another context in which factor sets arise. These also provide a nice connection between group cohomology and the Galois theory you studied earlier. Let K be a finite Galois extension of a field F with Galois group G. Let $f = \{a_{\sigma,\tau}\}_{\sigma,\tau \in G}$ be a normalized factor set of G with values in K^*. Let B_f be the vector space over K with basis u_σ for $\sigma \in G$. Define a ring structure on B_f via $u_\sigma \alpha = \sigma(\alpha) u_\sigma, u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$ for $\sigma, \tau \in G, \alpha \in K$. We call B_f a crossed product algebra for the factor set f (and G and K^*); see DF, p. 833. Note that if f is the constant function 1, then B_f is a kind of twisted analogue of the group algebra KG, where K no longer commutes with G but instead elements of G move past elements of K via the Galois group action. In general, u_1 is the multiplicative identity for B_f, since f is normalized.

If the factor set f is replaced by another one f' lying in the same cohomology class, then it is easy to check that the resulting algebra $B_{f'}$ is isomorphic to B_f by a map which is the identity on K^*. Thus there is a bijection between elements of $H^2(G, K^*)$ and K-isomorphism classes of crossed product algebras over F containing K. It is easy to see that the center of any algebra B_f is exactly $Fu_1 \cong F$: any combination $\sum_{\sigma \in G} \alpha_\sigma u_\sigma$ commutes with K if and only if $a_\sigma = 0$ for all $\sigma \neq 1$ (since only $\sigma = 1 \in G$ fixes all of K) and then ku_1 commutes with all u_σ if and only if $k \in K$ lies in F. Similarly, it is easy to see that B_f has no nonzero proper two-sided ideals: given a sum $s = \sum \alpha_\sigma u_\sigma$ with as few nonzero terms as possible lying in a proper nonzero ideal I, by replacing s with $s\beta - \beta s$ for suitable $\beta \in K$ we get a sum with fewer nonzero terms in I, unless s has only one term; but any single term $a\sigma u_\sigma$ is a unit in B_f. Thus B_f is indeed what is called a central simple algebra over F ("simple" meaning that it has no nonzero proper two-sided ideals).

Given any finite-dimensional central simple algebra A over a field F, A has a maximal proper left ideal L (that is, one of maximal dimension), so that A/L is an irreducible left A-module. M. Letting D be the ring of endomorphisms of M commuting with the A action, we see from Schur's Lemma (the version you proved in HW some weeks back) that D is a division ring containing F as a subfield. By the same reasoning that we used to determine the structure of the group algebra KG for G a finite group and K an algebraically closed field of characteristic not dividing the order of G, we deduce that $A \cong M_n(D)$, the ring of all $n \times n$ matrices over D (just one such ring rather than the direct sum of two or more since A is simple). It follows that the dimensions of both A and D over F are squares, at least in the special case $A = B_f$ for some factor set f (and it turns out in general).

You have already seen an example of such an algebra A, namely the ring H of quaternions (which is central simple over the real field $\mathbb{R}$). Note that H is a crossed product algebra: it contains a copy of the unique proper finite extension of $\mathbb{R}$, namely $\mathbb{C}$, and it contains an element j acting on $\mathbb{C}$ by complex conjugation, the unique nontrivial element of the Galois group of $\mathbb{C}$ over $\mathbb{R}$ (so that $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}$). The corresponding factor set f has $f(1,1) = f(1,j) = f(j,1) = 1, f(j,j) = -1$. There is no need here to take $\mathbb{R}$ as the base field; we could just as easily have started with $\mathbb{Q}$ and its Galois extension $\mathbb{Q}[i]$. In fact, given any $a, b \in \mathbb{Q}^*$ with a not a square in $\mathbb{Q}^*$, we could adjoin elements x, y to $\mathbb{Q}$ to make a central simple $\mathbb{Q}$-algebra $\mathbb{Q}_{a,b}$ with the defining relations $x^2 = a, y^2 = b, xy = -yx$. Every such algebra is either a division ring or isomorphic to the ring $M_2(\mathbb{Q})$ of 2×2 matrices over $\mathbb{Q}$.

In general, for any field F, we introduce an equivalence relation on central simple algebras A over F, which enables us to put a group structure on the set of such algebras. We have seen that $A \cong M_n((D)$ for some division ring D with center F (and D turns out to be unique); we decree that two such algebras A, A' are equivalent if both are isomorphic to a matrix ring over the same division ring D. The set of equivalence classes $[A]$ of central simple algebras A over F is then called the *Brauer group* $\text{Br}(F)$ of F.

Multiplication in this group is defined as follows. Given two central simple algebras A, B over the same F, the tensor product $C = A \otimes_F B$ acquires a multiplication via the recipe $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for $a, a', \in A, b, b' \in B$. One checks immediately that the recipe is compatible with the defining relations of the tensor product; it is also not difficult to see that C is central simple over F. This multiplication is well defined on equivalence classes. It turns out that the product $[B_f][B_g]$ of the classes of two crossed product algebras relative to the same F and K is just $[B_{fg}]$, the class corresponding to the product of the factor sets f, and g.

Of course $[F]$ is the identity element of $\mathrm{Br}(F)$. If $[A]$ lies in the Brauer group its inverse is the class $[A']$ of $A' = A^{\mathrm{opp}}$, where A' is defined to be A as an additive group but with reversed multiplication, so that ab in A' equals ba in A. Then $A \otimes_F A' \cong M_n(F)$, where $n = \dim_F A$, so that $[A']$ is indeed the inverse of $[A]$. The crossed product B_1 corresponding to the trivial factor set 1 is then isomorphic to a matrix ring over F, so that addition in $H^2(G, K^*)$ is compatible with multiplication in the Brauer group.

We conclude with a couple of examples. The Brauer group of $\mathbb{C}$, or any algebraically closed field, is trivial, since $\mathbb{C}$ does not admit any proper finite extension that is a division ring. The Brauer group of $\mathbb{R}$ has just two elements, namely the classes $[\mathbb{R}]$ and $[\mathbb{H}]$. The Brauer group of a finite field F is trivial, since there was a homework problem last quarter showing that a finite division ring is commutative, so that the only central simple algebras over F are matrix rings over F. The Brauer group of $\mathbb{Q}$ is huge, incorporating as it does the cohomology groups $H^2(G, K^*)$ for every finite Galois group G of an extension K of $\mathbb{Q}$ (which is conjecturally every finite group).