Lecture 5-27: Review, part 1

May 27, 2024

We will spend the rest of the course reviewing its content. The final exam will not cover any material outside of Dummit and Foote.

Let me begin with the general advice to concentrate in your studying on what is true, not why it is true. This may sound like heresy, coming from a math teacher, but the fact is that many students get so caught up in the details of particular proofs that they lose sight of what the proofs are about. Try to get a sense first of how the basic objects of study in the course behave and how they relate to each other, always keeping an eye out for simple consequences of powerful results. I won't ask you to reproduce the details of any proof in class.

The course began with an account of the rational canonical form for square matrices M over a field k, which asserts that M is similar to a block diagonal matrix in which the blocks are companion matrices $C(q_i)$ associated to powers $q_i = p_i^{n_i}$ of irreducible polynomials p_i over k. Here $C(q_i)$ has ones on the subdiagonal just below the main diagonal; the negatives of the coefficients of q_i in the rightmost column, apart from the leading coefficient; and zeroes elsewhere.

If k contains all the eigenvalues of M, then we can replace the $C(q_i)$ by Jordan blocks $J(\lambda_i)$, where the λ_i are the eigenvalues of M and $J(\lambda_i)$ has all diagonal entries equal to λ_i, ones above the diagonal, and zeroes elsewhere. If M is in Jordan form, then its minimal polynomial is the product $\prod_i (x - \lambda_i)^{m_i}$, where the λ_i run over the eigenvalues of M and m_i is the largest size of any Jordan block with eigenvalue λ_i. The characteristic polynomial of M is $\prod_i (x - \lambda_i)^{s_i}$, where s_i is the sum of the sizes of the blocks with eigenvalue λ_i.

Next I turned to representations of finite groups, that is, homomorphisms from a finite group G into $X = GL(V)$, the general linear group of invertible linear transformations π from a finite-dimensional vector space V over a fixed field k to itself. Two representations $\pi, \pi' : G \to X$ are equivalent if and only if they are conjugate, so that there is $x \in X$ with $\pi' = x\pi x^{-1}$. More generally, if V, W are vector spaces over the same field k that are isomorphic as G-modules, so that there is a linear isomorphism $\mu : V \to W$ commuting with the actions of G on V and W, then the corresponding representations π and π' are equivalent.

The nicest kind of representations of G occur over fields k that are algebraically closed and of characteristic not dividing the order of G. Here any (finite-dimensional) G-module V is the direct sum $\oplus_i V_i$ of irreducible submodules V_i (even if k is not algebraically closed). There are only finitely many irreducible G-modules up to equivalence; more precisely, there are as many such modules as there are conjugacy classes in G (though there is no canonical bijection between irreducible modules and conjugacy classes). In particular, all irreducible representations have dimension 1 if and only if G is abelian.

To prove these last two results I proved a structural result about the group algebra kG, defined to be the set of all formal linear combinations $\sum_{g \in G} k_g g$, with the coefficients $k_g \in k$. I showed that KG is isomorphic to a direct sum $\oplus_i M_{n_i}(k)$, where $M_{n_i}(k)$ denotes the ring of all $n_i \times n_i$ matrices over k; then there is one irreducible representation k^{n_i} of G for each term $M_{n_i}(k)$ in the direct sum. As a consequence, the sum $\sum_i n_i^2$ of the squares of the n_i equals the order $|G|$ of G.

In case k is the complex field $\mathbb{C}$, I got a better handle on a representation π by replacing the matrix $\pi(g)$ for $g \in G$ by a single number, namely its trace $\chi(g) = \text{tr } \pi(g)$. This is the sum of the eigenvalues of $\pi(g)$, each of which is a root of 1 in $\mathbb{C}$. For characters χ_i, χ_j of inequivalent irreducible representations, one then has the first Schur orthogonality relation, which states that $\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = 0$, while $\sum_{g \in G} \chi(g)\overline{\chi(g)} = |G|$, the order of G. The second orthogonality relation asserts that $\sum_i \chi(g)\overline{\chi_j(h)} = 0$ if $g, h \in G$ are not conjugate, where the sum runs over the irreducible characters χ_i of G. If g, h are conjugate, then $\sum_i \chi(g)\overline{\chi_i(h)} = \frac{|G|}{|C_g|}$, where C_g denotes the conjugacy class of g and $|C_g|$ its cardinality. Thus the irreducible characters provide an orthonormal basis for the space of complex (conjugacy) class functions on G.

Any character value $\chi(g)$ is an algebraic integer; that is, it is integral over $\mathbb{Z}$ (so that it satisfies a monic polynomial with integer coefficients). As a consequence, the degree n_i of any irreducible representation of G divides $|G|$. In particular, if G is a p-group, having order a power p^n of a prime p, then all of its irreducible representations have degrees that are powers of p as well.

Given a subgroup H of a finite group G and an irreducible H-module V_H over k, one has the induced G-module $V_G = \mathrm{Ind}_H^G(V_H) = kG \otimes_{kH} V_H$. Its character χ_G has $\chi_G(g) = \frac{1}{|H|} \sum_{x \in G, x^{-1}gx \in H} \chi_H(x^{-1}gx)$. The multiplicity of any irreducible G-module M_G in V_G equals the multiplicity of V_H in the restriction of M_G to H.

Using irreducible characters I showed that any finite group having a nonidentity conjugacy class C of size a power of a prime p has a nontrivial normal subgroup. Using induced characters I also showed that any finite group G having a subgroup H such that $x^{-1}HX \cap H = \{1\}$ for any $x \notin H$ also has a normal subgroup N such that $G = NH, H \cong G/N$.