Lecture 5-20: The prime spectrum of a ring

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We generalize the Zariski topology to an arbitrary Noetherian ring R in such a way that we add more points to affine space $\mathbf{A}^n$ and give it a richer structure.

In an attempt to generalize the Zariski topology from coordinate rings $k[V]$ of varieties to general Noetherian rings we might start with the set of maximal ideals of R. This set however does not have good functorial properties: the inverse image $f^{-1}(M)$ of a maximal ideal $M \subset S$ under a ring homomorphism $f : R \to S$ need not be maximal in R (think of the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ and the 0 ideal). The inverse image $P = f^{-1}(Q)$ of a *prime* ideal Q of S is prime in R, however, since the quotient R/P is a subring of S/Q and so cannot have zero divisors if S/Q does not. Accordingly, we start with the set Spec R of prime ideals in R, calling this the prime spectrum of R (DF, p. 731).

We define the Zariski topology on Spec R by decreeing that the closed sets are the sets $\mathcal{Z}(I)$ of prime ideals containing a fixed ideal I of R (DF, p. 733). Clearly any prime ideal P containing an ideal I also contains its radical $\sqrt{I}$, since a power $x^n \in P$ if and only if $x \in P$. Conversely, the radical $\sqrt{I}$ of I is exactly the intersection $\mathcal{I}(I) = \cap\mathcal{Z}(I)$ of all prime ideals P containing I (DF, Proposition 12, p. 674). To prove this, let $x \in R$ be such that $x^n \notin I$ for all n and let $\mathcal{S}$ be the collection of ideals in R excluding all powers of x. Then $\mathcal{S}$ is not empty since $I \in \mathcal{S}$; let P be a maximal element of $\mathcal{S}$ under inclusion. If $y, z \notin P$ then $P + (y), P + (z)$ are strictly larger than P and so they must contain powers x^r, x^s of x; but then $P + (yz)$ contains x^{r+s}, forcing $yz \notin P$, whence P is prime.

Hence any radical ideal I is the intersection $\mathcal{I}(\mathcal{P})$ of the set $\mathcal{P}$ of prime ideals containing I, and conversely any intersection of prime ideals is radical. We deduce

Proposition 53, DF, p. 732

There are inclusion-revsersing bijections $\mathcal{Z}(I) \mapsto \mathcal{I}(I), I \mapsto \mathcal{Z}(I)$ between the sets of Zariski closed subsets of Spec R and of radical ideals of R.

We also see that finite unions and arbitrary intersections of Zariski closed sets are closed, since $\mathcal{Z}(IJ) = \mathcal{Z}(I) \cup \mathcal{Z}(J)$ and $\mathcal{Z}(\sum_i I_i) = \cap_i \mathcal{Z}(I_i)$.

As with algebraic sets, one also readily checks that a Zariski closed subset $\mathcal{Z}(I)$ of Spec R is irreducible (not the union of two proper closed subsets) if and only if its radical ideal $\mathcal{I}(I)$ is prime.

If $R = k[V]$ is the coordinate ring of an algebraic set V, then the prime spectrum $X =$ Spec R contains the set mSpec $k[V]$ of maximal ideals of $k[V]$, which corresponds bijectively to V. For every maximal ideal M, the singleton set $\{M\} = \mathcal{Z}(M)$ and so is closed; by abuse of language we call M a closed point in X (DF, p. 733). Now however we have additional points in X that are not closed. For example, if V is a variety, then the closure of $\{0\}$ is all of X. We call this ideal a generic point.

Returning now to a general Noetherian ring R, let $f : R \to S$ be a homomorphism of rings. Then f induces a map $f^* : \text{Spec } S \to \text{Spec } R$ sending a prime ideal Q of S to the prime ideal $f^{-1}(Q)$ of R. It is easy to check that the inverse image under f^* of the Zariski closed set $\mathcal{Z}(I)$ for an ideal I of R is the closed set $\mathcal{Z}(J)$, J, the ideal of S generated by $f(I)$, whence we get

Proposition 35, DF, p. 734

With notation as above the induced map f^* from Spec S to Spec R is continuous with respect to the Zariski topology.

Just as an arbitrary Zariski-continuous map from one variety V to another one W need not be a morphism and so need not correspond to a ring homomorphism, there is no reason to expect an arbitrary continuous map from Spec S to Spec R to arise from a ring homomorphism from R to S.

Now let $f \in R$. Corresponding to the Zariski principal open set $D_f \subset \mathbf{A}^m$ of nonzeroes of a polynomial $f \in P_m$ we have the basic open set X_f of ideals in $X = \text{Spec } R$ not containing f. Since a Zariski closed set $\mathcal{Z}(I)$ correpsonds to a finitely generated ideal $I = (f_1, \ldots, f_m)$, we see that the X_f form a basis for the Zariski topology and in fact every Zariski open set is a finite union of X_f (DF, Proposition 56, p. 738).

We now define the analogue of regular function for a Noetherian ring.

Definition; DF, p. 739

Let U be a nonempty open subset of Spec R. Define $\mathcal{O}(U)$ to be the set of functions $s : U \to \sqcup_{Q \in U} R_Q$ from U to the disjoint union of the localizations R_Q for $Q \in U$ such that

- $s(Q) \in R_Q$ for $Q \in U$, and
- for every $P \in U$ there is a basic open neighborhood $X_f \subseteq U$ of P in U and an element $\frac{a}{f^n}$ in the localization R_f defining s on X_f, so that $s(Q) = \frac{a}{f^n}$ for $Q \in X_f$.

It is easy to check that each $\mathcal{O}(U)$ is closed under addition and multiplication and that there is a natural restriction map from $\mathcal{O}(U)$ to $\mathcal{O}(U')$ whenever U' is a nonempty open subset of U.

Definition; DF, p. 740

With notation as above, the collection of rings $\mathcal{O}(U)$ as U runs over the nonempty Zariski open subsets of $X = \text{Spec } R$ together with the restriction maps $\mathcal{O}(U) \to \mathcal{O}(U')$ for $U' \subseteq U$ is called the *structure sheaf* of X and is denoted $\mathcal{O}$ or $\mathcal{O}_X$. The elements s of $\mathcal{O}(U)$ are called *sections of* $\mathcal{O}$ *over* U. The elements of $\mathcal{O}(X)$ are called *global sections*.

A straightforward argument then proves

Proposition 57, DF, p. 740

If $X = \text{Spec } R$ has structure sheaf $\mathcal{O}$, then the global sections of $\mathcal{O}$ identify naturally with the elements of R. More generally, if X_f is a basic open set, then $\mathcal{O}(X_f)$ identifies with the localization R_f.

Definition; DF, p. 741

If $P \in X$ then the localization R_P of R at P is called the *stalk of* $\mathcal{O}$ *at* P.

There is a nice picture of this on p. 742. The pair $(X, \mathcal{O}_X)$ with $X = \text{Spec } R$ is called an affine scheme. The stalk $\mathcal{O}_P = R_P$ of $\mathcal{O}$ at P may be viewed as a direct limit of rings $\mathcal{O}(U)$ as U runs through the open sets containing P.

We have mentioned that continuous maps from $X = \text{Spec } S$ to $X' = \text{Spec } R$ do not generally correspond to ring homomorphisms from R to S; but if the continuous map is paired with compatible ring homomorphisms from sections over open subsets for X to corresponding sections for X' commuting with restriction maps and taking stalks to stalks, then it does correspond to a ring homomorphism. In this way we get a bijection between ring homomorphisms and what are called morphisms of affine schemes.

Finally, a locally ringed space is a topological space X equipped with a ring $\mathcal{O}(U)$ attached to every nonempty open subset U with a compatible set of homomorphisms from $\mathcal{O}(U)$ to $\mathcal{O}(U')$ if $U' \subseteq U$ and with local conditions on the sections, such that the stalks (direct limits of $\mathcal{O}(U)$ as U runs through the open sets containing a fixed point $P \in X$) are local rings. A general scheme is a locally ringed space in which each point lies in a neighborhood isomorphic to an affine scheme (with some compatibility conditions between such neighborhoods). These are the algebro-geometric analogues of differentiable manifolds and play a fundamental role in modern algebraic geometry.