Lecture 5-17: Nonsingular points, schemes, and projective space

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We give a first example of a scheme, namely a quasi-affine variety (open subset of a variety), and wrap up the treatment of nonsingular points on varieties from last time. We then look at another example of a scheme, namely projective space.

Let U be a nonempty open subset of $\mathbf{A}^n$. Then U need not have the structure of a variety, but it is a finite union of affine sets (isomorphic to varieties). To see this, recall that U is the complement of a closed set $\mathcal{Z}(f_1, \ldots, f_m)$ and thus a finite union of principal open sets V_{f_i}, each consisting of the points where a single polynomial f_i does not vanish. Then V_{f_i} is isomorphic to the subvariety of $\mathbf{A}^{n+1}$ defined by the equation $f_i(x_1, \ldots, x_n)x_{n+1} = 1$ and the polynomial $f(x_1, \ldots, x_n)x_{n+1}$ is easily seen to be irreducible. Consequently $\mathbf{A}^n$ is quasicompact in the sense that every open cover admits a finite subcover; we call this property "quasicompactness" rather than compactness because $\mathbf{A}^n$ is not a Hausdorff space.

In the same way, any nonempty open subset U of a variety V can be written as a finite union of varieties (each irreducible since its coordinate ring is a localization of the coordinate ring of V). Such a subset is called a quasi-affine variety. Every point in U has a neighborhood that is affine. A topological space with this property is a special case of a scheme; we will later generalize this term to a wider class of spaces.

For now we introduce another notion (which has no counterpart for differentiable manifolds). Two varieties V, W are called birational if their function fields $k(V), k(W)$ are isomorphic as extensions of k. Whenever this happens the isomorphism from $k(V)$ to $k(W)$ maps the coordinate ring $k[V]$ onto a subring of $k(W)$ finitely generated as a k-algebra. This need not correspond to a map defined on all of W, but it does correspond to an isomorphism ϕ from an open subset W' (defined by the nonvanishing of a suitable polynomial) to an open subset V' of V (in the sense that every point of W' has an affine neighborhood on which ϕ restricts to an isomorphism of varieties).

We call ϕ a rational map from W to V (rational in the sense that it is not necessarily defined on all of W; these are the higher-dimensional analogues of the regular maps defined earlier). Conversely, it is not difficult to see that two varieties V, W admitting nonempty isomorphic open subsets V', W' in the above sense are birational.

Now we can pick up from where we left off last time. Let V be a variety of dimension d. We know that the tangent space $T_v V$ of V at a nonsingular point v has a constant dimension; we would like to show that this dimension is d. To this end, recall by Noether normalization that the coordinate ring $k[V]$ of V is a finitely generated integral extension of a polynomial ring $k[y_1, \ldots, y_d]$. Thus the function field $k(V)$ is a finite extension of $k(y_1, \ldots, y_d)$; changing the y_i if necessary we may assume that it is a separable extension of this field (or just assume for simplicity that the characteristic of k is 0, so that separability is automatic). By the theorem of the primitive element, $k(V)$ is then generated by a single element y over $k(y_1, \ldots, y_d)$, say with minimal polynomial q.

Writing q as a polynomial in $y = y_{d+1}$ with coefficients in $k(y_1, \ldots, y_d)$, let f be the product of the denominators of these coefficients. Replacing y by $y' = \frac{y}{f}$, we find that the minimal polynomial q' of y' has coefficients that are polynomials in $y_1, \ldots, y_d$, so that it is a polynomial in $k[y_1, \ldots, y_d], y_{d+1}$. The hypersurface H in $\mathbf{A}^{d+1}$ defined by the equation $q' = 0$ is then birational to the original variety V, so has an open subset H' isomorphic to an open subset V' of V. The tangent space at any point h of H', thanks to its realization as the dual of the quotient $\mathfrak{m}_h/\mathfrak{m}_h^2$, where $\mathfrak{m}_h$ is the maximal ideal of the local ring $\mathcal{O}_{H',h}$ of H' at h, is the same as that of H.

But now H has a tangent space of dimension $d+1-1=d$ at any point where the gradient of its defining polynomial q' is not 0. There is an open subset of such points which is clearly nonempty in characteristic 0. In characteristic $p>0$ the only way that it could be empty is if q' were a polynomial in the pth powers $x_1^p, \ldots, x_{d+1}^p$ of its variables $x_1, \ldots, x_{d+1}$. But if so we could take pth roots of its coefficients in k (since k is algebraically closed) to write q' as the pth power of another polynomial, contradicting the irreducibility of q'. We finally conclude that the set of nonsingular points of V is nonempty and open, the tangent space at any of them having dimension d, while the set of its singular points is a proper closed subvariety (see DF, p. 725). For example, the only singular point of our old friend the variety in $\mathbf{A}^2$ defined by $x^3-y^2=0$ is $(0,0)$. The tangent space at this point is two-dimensional.

Now we broaden our horizons, looking at schemes that are not quasi-affine. The most common examples of such schemes in practice are projective varieties. Start by removing the origin $\vec{0} = (0, \ldots, 0)$ from affine $(n+1)$-space $\mathbf{A}^{n+1}$. Introduce an equivalence relation on this space, declaring that $(a_1, \ldots, a_{n+1}) \sim (xa_1, \ldots, xa_{n+1})$ whenever $x \in k^*$. The resulting quotient space $\mathbf{P}^n$ of equivalence classes is called projective n-space, with a point in it often denoted by $[a_1, \ldots, a_{n+1}]$. The brackets serve as a reminder of the equivalence relation.

No nonconstant polynomial $p \in P_{n+1}$ takes a well-defined value on every point in $\mathbf{P}^n$; but if p is homogeneous, say of degree d, then $p(a_1, \ldots, a_{n+1}) = 0$ if and only if $p(xa_1, \ldots, xa_{n+1}) = 0$, for all $x, a_1, \ldots, a_{n+1} \in k$. Hence the zero locus $\mathcal{Z}(p)$ of any such polynomial in $\mathbf{P}^n$, or more generally the zero locus $\mathcal{Z}(p_1, \ldots, p_m)$ of a finite set of homogeneous polynomials in P_{n+1} in $\mathbf{P}^n$, not necessarily of the same degree, is well defined.

In fact the zero locus $\mathcal{Z}(f_1, \ldots, f_m)$ in affine space $\mathbf{A}^{n+1}$ of any set $f_1, \ldots, f_m$ of polynomials in P_{n+1}, is a union of lines through the origin, or equivalently of $\sim$ equivalence classes and $\vec{0}$, if and only if the ideal I generated by the f_i is homogeneous. That is, it contains the homogeneous part p_d of degree d of any of its elements p, for all d. As a quotient topological space of $\mathbf{A}^{n+1}$ minus the origin, $\mathbf{P}^n$ inherits a topology induced by the Zariski topology. The closed sets in this topology are the zero loci in $\mathbf{P}^n$ of finite sets of homogeneous polynomials in P_{n+1}.

From the Nullstellensatz we therefore deduce

Theorem: homogeneous Nullstellensatz

There is an inclusion-reversing bijection between radical homogeneous ideals of P_{n+1} other than $M = (x_1, \ldots, x_{n+1})$ and algebraic (Zariski closed) subsets of $\mathbf{P}^n$.

Here the ideal M has to be omitted from the correspondence as its only common zero in $\mathbf{A}^{n+1}$ is the deleted point $\vec{0}$; for this reason it is sometimes called the *irrelevant ideal*.

The notion of irreducibility then carries over from the Zariski topology on $\mathbf{A}^{n+1}$; irreducible algebraic subsets of $\mathbf{P}^n$ are called projective varieties; they correspond to homogeneous prime ideals in P_{n+1}. The notion of dimension using chains of (now homogeneous) prime ideals then also carries over; since every chain $I_0 \subset \cdots \subset I_d$ of distinct homogeneous prime ideals in P_{n+1} other than M can be lengthened by adding M at the end, we see that the dimension of the projective variety P defined by the homogeneous polynomials $f_1, \ldots, f_m \in P_{n+1}$ is one less than the dimension d of the affine variety V defined by the same polynomials, provided that $d \geq 1$.

In order to see that $\mathbf{P}^n$ and its closed subvarieties are schemes, we need to decompose this space into affine subvarieties. To this end, observe for all indices i between 1 and $n+1$ that the set P_i of points $[x_1, \ldots, x_{n+1}] \in \mathbf{P}^n$ with $x_i \neq 0$ may be naturally identified with $\mathbf{A}^n$ via the map $(y_1, \ldots, y_n) \mapsto [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]$, since $[x_1, \ldots, x_{n+1}] \sim [\frac{x_1}{x_i}, \ldots, 1, \ldots \frac{x_{n+1}}{x_i}]$. In this way we get a large collection of regular functions on $\mathbf{P}^n$, or more generally on any projective variety, extending the definition of regular function in the obvious way from varieties to schemes. The key difference from the affine setting is that projective varieties, unlike quasi-affine ones, do not sit in larger affine varieties. In fact, it turns out that the only regular functions defined at all points of a projective variety are constants. In order to understand projective varieties, we have to look at them from both a local and a global point of view.