Lecture 4-26: Group cohomology

April 26, 2024

I will pick up from the definition of the cohomology groups $H^n(G, A)$ of a finite group G with coefficients in a G-module A, given last time. Such groups, like complex representations, are interesting in their own right; they also yield important purely group-theoretic information.

Note that G-modules A have a very different flavor from representations of G over a field K. In particular, since free $\mathbb{Z}$-modules are never semisimple our focus will not be on irreducible modules but rather on the way that the submodule A^G of G-invariants in A (consisting of all $a \in A$ fixed by every $g \in G$) sits inside A.

As mentioned last time, we define $H^n(G, A)$, the nth cohomology group of G with coefficients in A, to be the Ext group $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module (on which every $g \in G$ acts trivially). See p. 800 of DF. Thus we can compute these groups using any convenient projective resolution of $\mathbb{Z}$. For example, if G is cyclic of order m with generator σ, then we have a resolution

$$\cdots \to \mathbb{Z}G \to \mathbb{Z}G \to \cdots \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

where the maps from $\mathbb{Z}G$ to itself alternate between multiplication by $\sigma - 1$ and multiplication by $N = 1 + \sigma + \cdots + \sigma^{n-1}$, and the map from $\mathbb{Z}G$ to $\mathbb{Z}$ is the augmentation map sending $\sum_G z_g g$ to $\sum z_g$.

Arguing as in the computation of $\mathrm{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D)$ on April 22, we see that

$$H^n(G,A) = \begin{cases} A^G & \text{if } n = 0 \\ A^G/NA & \text{if } n \text{ even}, n \geq 2 \\ {}_NA/(\sigma - 1)A & \text{if } n \text{ odd}, n \geq 1 \end{cases}$$

where ${}_NA$ is the subgroup of A sent to 0 by N. In particular, if G acts trivially on A, then $Na = ma$ for all $a \in A$, so that $H^0(G,A) = A$, $H^n(G,A) = A/mA$ for even $n \geq 2$, while $H^n(G,A) = {}_mA$, the elements of A sent to 0 by m, for n odd.

For more complicated groups G the standard projective resolution of $\mathbb{Z}$ arising in this week's HW (called the bar resolution) gives a uniform recipe for computing $H^n(G, A)$. Since this is a free resolution it is easy to compute $\hom_{\mathbb{Z}G}(F_n, A)$ for all A. We are then led to the following direct definition of $H^n(G, A)$ (making no reference to Ext groups or projective resolutions).

Denote by $C^n(G, A)$ the set of all maps from G^n to A for $n > 0$ and set $C^0(G, A) = A$. The elements of $C^n(G, A)$ are called *homogeneous n-cochains of G (with values in A)*. For $n \geq 0$ define d_n, the *nth coboundary homomorphism* from $C^n(G, A)$ to $C^{n+1}(G, A)$ via

$$\begin{aligned} d_n(f)(g_1, \ldots, g_{n+1}) = & \, g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

One can then check (with some effort) that the composition $d_n d_{n-1}$ is 0, so that it makes sense to form the quotient $Z^n(G, A)/B^n(G, A)$ of the kernel $Z^n(G, A)$ of d_n (the *n-cocycles*) by the image $B^n(G, A)$ of d_{n-1} (the *n-coboundaries*). This quotient is then equal to $H^n(G, A)$.

This formulation is frankly not all that useful for computing $H^n(G, A)$; the point of it is instead to interpret these groups. We will see that in particular we can use the first and second cohomology groups to compute group-theoretic information. For now we just observe that if $mA = 0$ for some integer m, then $mZ^n(G, A) = 0$ for all n, so that $mH^n(G, A) = 0$ also. Corresponding to the long exact sequence for Ext groups arising from a short exact sequence of R-modules we get

Theorem 21, DF, p. 802

Given a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules we get a long exact sequence

$$0 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to \cdots$$

of abelian groups.

Here the connecting homomorphisms $\delta_n : H^n(G, C) \to H^{n+1}(G, A)$ are those of the long exact sequence for Ext.

Next we introduce the analogue of the induced representation construction.

Definition, DF, p. 803

If H is a subgroup of G and A is an H-module, then we define the *coinduced module* to be $M = M_H^G(A) = \hom_{\mathbb{Z}H}(\mathbb{Z}G, A)$, where the action of G is given by $g \cdot f(x) = f(xg)$ if $f \in M, g \in G$.

We also have $M \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ (since G is finite; but see next week's HW assignment for a counterexample in general), so M is also often called the module induced from A.

Proposition: Shapiro's Lemma, DF, p. 804

For any subgroup H of G and H-module A we have $H^n(G, M_H^G(A)) \cong H^n(H, A)$.

Proof.

Set $M = M_H^G(A)$. The bar resolution of $\mathbb{Z}$ over G is a free resolution which remains free over H, since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Taking G-homomorphisms into M we get a cochain complex whose cohomology groups are the groups $H^n(G.M)$. But now for any G-module P there is a natural isomorphism $\Phi : \hom_{\mathbb{Z}G}(P, \hom_{\mathbb{Z}H}(\mathbb{Z}G, A)) \cong \hom_{\mathbb{Z}H}(P, A)$ of abelian groups defined by $\Phi(f)(p) = f(p)(1)$. Since this isomorphism commutes with the coboundary maps it induces an isomorphism between the corresponding cohomology groups, as desired. □

This is the analogue of Frobenius Reciprocity for coinduced modules.