Lecture 4-24: Ext groups, module extensions, and group cohomology

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We will show that Ext groups are well defined (independent of the choice of projective resolution) and derive some long exact sequences. We then define group cohomology, using a particular projective resolution of a trivial G-module.

We first prove

Proposition (DF, p. 782)

Using the same notation as last time, given a homomorphism $f : A \to A'$ of R-modules, the induced maps $\phi_n : \mathrm{Ext}^n_R(A', D) \to \mathrm{Ext}^n_R(A, D)$ depend only on f, not on the choice of lifts f_n used to define these maps.

Proof.

For this it is enough to show that if f is the zero map, then the induced maps are all zero as well. If $f = 0$ then by projectivity of the P_i one can inductively define homomorphisms $s_n : P_n \to P'_{n+1}$ such that $f_n = d'_{n+1}s_n + s_{n-1}d_n$. The collection of maps $\{s_n\}$ is called a chain homotopy between the chain homomorphism given by the f_n and the zero homomorphism. Taking homomorphisms into D we get the diagram used to determine the ϕ_n, together with additional diagonal arrows representing a chain homotopy between cochain complex homomorphisms. Using these diagonal maps, it is easy to see that any element in $\hom_R(P'_n, D)$ representing a coset in $\mathrm{Ext}^n_R(A', D)$ maps to the zero coset in $\mathrm{Ext}^n_R(A, D)$, as claimed. □

Now we can finally prove the independence result promised last time.

Theorem (DF, p. 782)

The groups $\operatorname{Ext}^n_R(A, D)$ depend only on A and D.

Indeed, given two projective resolutions $\{P_i\}, \{P'_i\}$ of A and the lifts f_n of the identity map $f : A \to A' = A$, the resulting maps ϕ_n from $\operatorname{Ext}^n_R(A, D)$ (computed via the P_i) and $\operatorname{Ext}^n_R(A', D)$ (computed via the P'_i) admit maps $\phi_n : \operatorname{Ext}^n_R(A', D) \to \operatorname{Ext}^n_R(A, D)$ such that the composites $\phi'_n \phi_n$ are induced from the identity map from A to A. These composites are also identity maps, by the previous result, and the assertion follows.

Given a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules and projective resolutions of L and N, say by $\{P_i\}$ and $\{P_i'\}$, respectively, it is not difficult to construct a projective resolution of M by $\{Q_i = P_i \oplus P_i'\}$ (DF p. 783) such that the resolutions of L, M, N fit into a short exact sequence of chain complexes. Taking homomorphisms into another R-module D and the corresponding long exact sequence in cohomology, we get a sequence

$$0 \to \hom_R(D, L) \to \hom_R(D, M) \to \hom_R(D, N) \xrightarrow{\gamma_0} \mathrm{Ext}_R^1(D, L)$$

$$\to \mathrm{Ext}_R^1(D, M) \to \mathrm{Ext}_R^1(D, N) \to \cdots$$

called the long exact sequence for Ext.

In particular, it follows from this sequence and the trivial projective resolution $\cdots \to 0 \to P \to P \to 0$ of any projective module P that an R-module P is projective if and only if $\mathrm{Ext}^1_R(P, B) = 0$ for all R-modules B, or if and only if $\mathrm{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$.

The notation Ext here stands for “extension”. An extension of an R-module N by another one L is a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules. Given two such extensions, say

$$0 \to L \to M \to N \to 0$$

$$0 \to L \to M' \to N \to 0$$

we say they are equivalent if there is an isomorphism $f : M \to M'$ making the above diagram commute when supplemented by the identity maps from L and N to themselves (DF p. 381).

For example, taking $R = \mathbb{Z}$ and letting p be a prime, there is just one equivalence class of extensions of $L = \mathbb{Z}/(p)$ by itself with middle term isomorphic to $E = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$, since the automorphism group of E, consisting of all 2×2 invertible matrices over $\mathbb{Z}/(p)$, is large enough to identify any two such extensions. On the other hand, there are $p - 1$ inequivalent extensions of L with middle term $E' = \mathbb{Z}/(p^2)$, for if we fix the embedding of L into E' sending the coset of i mod p to that of pi mod p^2, then there are $p - 1$ inequivalent ways to map E' onto L so as to complete the short exact sequence. These are indexed by the nonzero elements i of L, the ith one sending the coset of x to that of ix mod p.

Thus there are in all p inequivalent extensions of L by itself. It is no coincidence that $\mathrm{Ext}^1_{\mathbb{Z}}(L, L)$ is cyclic of order p (by a previous calculation), as it is known in general that there is a bijection between $\mathrm{Ext}^1_R(N, L)$ and equivalence classes of extensions of N by L (Theorem 12, DF, p. 787). In this bijection the zero element of the Ext group corresponds to the split extension $L \oplus N$. More generally, for any $n \geq 1$ there is a notion of equivalence class of n-extensions of N by L, that is, equivalence classes of exact sequences $0 \to L \to M_n \cdots \to M_1 \to N \to 0$ and a bijection due to Yoneda between the nth Ext group $\mathrm{Ext}^n_R(N, L)$ and such extensions. There is also the structure of an additive group on equivalence classes of extensions corresponding to the additive group structure on Ext^1.

We conclude with a quick definition of group cohomology, so that you can get started on one of the problems in this week's homework. Given a finite group G, let $F = \mathbb{Z}G$ be its integral group ring, defined in the same way as the group algebra $\mathbb{C}G$, except that integers rather than complex numbers are used as coefficients. Then an F-module is just an abelian group A (or $\mathbb{Z}$-module) on which G acts by automorphisms. There is a standard resolution of $\mathbb{Z}$, the trivial G-module, with nth term $F_n = \otimes^{n+1} F$, the $(n+1)$st tensor power of F, which is a free F-module with basis $1 \otimes g_1 \otimes \cdots \otimes g_n$ as the g_i range over G. Here G acts by multiplication on the leftmost copy of F.

Denoting this basis element by $(g_1, \ldots, g_n)$ we have the following recipe for the nth coboundary operator d_n on F_n:

$$
\begin{aligned}
d_n(g_1, \ldots, g_n) = g_1 \cdot (g_2, \ldots, g_n) \\
+ \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_n) \\
+ (-1)^n (g_1, \ldots, g_{n-1})
\end{aligned}
$$

In Exercise 1 on p. 809 of this week's HW you will use a *contracting homomorphism* and the *augmentation map* defined there to show that the sequence $\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ is exact, so that this is indeed a projective resolution of $\mathbb{Z}$, which can be used to compute $\mathrm{Ext}_F^n(\mathbb{Z}, A)$. This last group is by definition the nth cohomology group $H^n(G, A)$. More to come!