Lecture 4-22: Homological algebra

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We turn now to Chapter 17 of Dummit and Foote, first studying the Ext groups attached to a possibly noncommutative ring R (with 1) and a left R-module M. Recall that M is called projective if given any short exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules, the induced sequence $0 \to \hom_R(M, N') \to \hom_R(M, N) \to \hom_R(M, N") \to 0$ of abelian groups is exact, or equivalently for any surjective R-module homomorphism $N \to N"$ the homomorphism $\hom_R(M, N) \to \hom_R(M, N")$ is also surjective. You have previously seen that M is projective if and only if it is a direct summand of a free R-module.

It turns out that if M is not projective over R, then its failure to be projective can be measured in a precise way. To do this we first need to define and study the general notions of chain and cochain complexes. A sequence $\mathcal{C}$ of abelian group homomorphisms $\cdots \to C_n \to \cdots \to C_0 \to 0$ with $d_n : C_n \to C_{n-1}$, is called a chain complex if the composition of any two successive maps is 0 (see DF p. 777). If instead $\mathcal{C}$ takes the form $0 \to C^0 \to C^1 \to \cdots \to C^n \to \cdots$, with $d_n : C^{n-1} \to C^n$, then we call it a cochain complex if the composition of two successive maps is 0. If $\mathcal{C}$ is a chain complex then its nth homology group $H_n(\mathcal{C})$ is the quotient $\ker d_n / \operatorname{im} d_{n+1}$, where im denotes the image of a map; if $\mathcal{C}$ is a cochain complex then its nth cohomology group $H^n(\mathcal{C})$ is the quotient $\ker d_{n+1} / \operatorname{im} d_n$. The cochain complex is exact if and only if its cohomology groups are trivial.

For convenience we will work exclusively with cochain complexes (following DF). The maps d_n are called coboundary maps and their kernels are called cocycles (for chain complexes they are called boundary maps and their kernels are called cycles). Given two complexes $\mathcal{A} = \{A^n\}, \mathcal{B} = \{B^n\}$, a homomorphism of chain complexes $\alpha : \mathcal{A} \to \mathcal{B}$ is a set of homomorphisms $\alpha_n : A^n \to B^n$ commuting with the coboundary maps. It is easy to check that such a homomorphism induces group homomorphisms from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$ for $n \geq 0$. A short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of cochain complexes is a pair of homomorphisms $\alpha : \mathcal{A} \to \mathcal{B}, \beta : \mathcal{B} \to \mathcal{C}$ such that the maps $0 \to A^n \to B^n \to C^n \to 0$ from a short exact sequence for all n. Then we have

Theorem: Long Exact Sequence in Cohomology (DF p. 778)

Given a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, there are maps $\delta_n : H^n(\mathcal{C}) \to H^{n+1}(\mathcal{A})$ such that the sequence $0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to H^1(\mathcal{A}) \to \cdots$ is exact.

You will work out the details of the proof in HW (DF Exercise 2, p. 791). The maps δ_n are called connecting homomorphisms. A consequence of this result is that if any two of the cochain complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are exact then so is the third.

Now we can address the problem of measuring non-projectivity precisely. Henceforth R will be a fixed ring, A a left R-module, and all maps will be R-module maps.

Definition: projective resolution (DF p. 779)

A projective resolution of A is an exact sequence $P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ of R-modules with the P_i projective; we denote by d_n the map $P_n \to P_{n-1}$ and by ϵ the map $P_0 \to A$.

Note that every module has a projective resolution, since free modules are projective: just a choose a free module P_0 surjecting onto A, say with kernel K_0, then a free module P_1 surjecting onto K_0, say with kernel K_1, and so on. If A is itself projective, then we have the very simple resolution $0 \to A \to A \to 0$; if A has a finite projective resolution (with only finitely many nonzero terms), then A is not too far from bring projective.

Now let A, D be two R-modules and $\{P_i\}_{i \geq 0}$ a projective resolution of A. Applying the contravariant functor $\hom(\cdot, D)$ to each term of the resolution, we get a cochain complex $0 \to \hom_R(A, D) \to \hom(P_0, D) \to \hom(P_1, D) \to \cdots$; denote again by $\epsilon, d_1, d_2, \ldots,$ the maps induced by the boundary maps $\epsilon, d_1, d_2, \ldots$ sending P_0 to A, P_1 to P_0, P_2 to P_1, and so on, in the resolution.

Definition: Ext groups (DF p. 779)

With notation as above, we define $\mathrm{Ext}_R^n(A, D)$ to be the quotient of the kernel of d_{n+1} by the image of d_n for $n \geq 1$, while $\mathrm{Ext}_R^0(A, D) = \ker d_1$. The Ext groups are the *higher derived functor* groups of the contravariant functor $\hom(\cdot, D)$. If $R = \mathbb{Z}$, then $\mathrm{Ext}_{\mathbb{Z}}^n(A, D)$ is denoted simply $\mathrm{Ext}^n(A, D)$.

We will see next time that these groups do not depend on the choice of projective resolution of A. For now we just note that this is true of Ext^0: since $\hom(\cdot, D)$ is left exact, the sequence $0 \to \hom_R(A, D) \to \hom_R(P_0, D) \to \hom_R(P_1, D)$ is exact, whence the kernel of d_1 in this sequence equals the image of ϵ, which is just $\hom_R(A, D)$: $\mathrm{Ext}^0_R(A, D) = \hom_R(A, D)$.

Example

Take $R = \mathbb{Z}$, $A = \mathbb{Z}/m\mathbb{Z}$, By the above observation we have $\mathrm{Ext}^0_{\mathbb{Z}}(A, D) \cong {}_mD$, where ${}_mD$ denotes the elements of D sent to 0 by m. We have a projective resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$, where the map from $\mathbb{Z}$ to itself is multiplication by m and the map from $\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ is the canonical one. Taking homomorphisms into D and applying the above definition we get $\mathrm{Ext}^0_{\mathbb{Z}}(A, D) \cong {}_mD$, $\mathrm{Ext}^1_{\mathbb{Z}}(A, D) \cong D/mD$, $\mathrm{Ext}^n_{\mathbb{Z}}(A, D) = 0$ for $n \geq 2$.

Example

On the other hand, if we replace R by $\mathbb{Z}/m\mathbb{Z}$ and A by $\mathbb{Z}/d\mathbb{Z}$, where d is a divisor of m, then we have the infinite projective resolution $\cdots\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$, where the maps from $\mathbb{Z}/m\mathbb{Z}$ to itself are alternately multiplication by d and multiplication by m/d. Taking homomorphisms into D, we now get $\mathrm{Ext}_R^0(A, D) \cong {}_dD$, $\mathrm{Ext}_R^n(A, D) \cong {}_{m/d}(D/dD)$ for n odd, and $\mathrm{Ext}_R^n(A, D) \cong {}_d(D/(m/d)D)$ for n even, $n \geq 2$. Thus the Ext groups depend on the base ring.

We now head toward the proof that the Ext groups are independent of the choice of projective resolution. First we show

Proposition (DF, p. 781)

Let $f : A \to A'$ be an R-module map and let $\{P_n\}, \{P'_n\}$ be projective resolutions of A, A', having boundary maps $\epsilon, d_1, d_2, \ldots$ and $\epsilon', d'_1, d'_2, \ldots$, respectively. Then there are lifts $f_i : P_i \to P'_i$ (together with f) making the obvious diagram commute.

This follows at once from the lifting property of projective modules. Thus we also get induced lifts $f, f_0, f_1, \ldots$ between the terms of the cochain complexes
$0 \to \hom_R(A, D) \to \hom_R(P_0, D) \to \cdots$ and
$0 \to \hom_R(A', D) \to \hom_R(P_0, D) \to \cdots$ and thereby induced maps on their cohomology groups.

Next time we will show that the maps $\phi_n : \mathrm{Ext}_R^n(A', D) \to \mathrm{Ext}_R^n(A, D)$ depend only on f, not on the choice of lifts f_n made above.