Lecture 4-19: Hook formula and symmetric functions

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Whole books are written and whole courses devoted to the representations of the symmetric group; while I cannot go into depth on this topic, there are couple of additional topics that I would like to mention. I will give a beautiful formula for the number of standard tableaux of a fixed shape and give an inkling of the many connections between the symmetric group and symmetric polynomials.

Given a Young diagram, each box in it corresponds to a hook, consisting of itself and all boxes either directly below it or directly to its right; its length is the number of boxes in it. For example, starting with the partition $\lambda = (3, 2, 1)$, I have labelled each box in the corresponding Young diagram by the length of its hook below.

$$\begin{array}{ccc} 5 & 3 & 1 \\ 3 & 1 & \\ 1 & & \end{array}$$

Then we have

Hook length formula

If $n = |\lambda|$, then the number of standard tableaux of shape λ equals $n!$ divided by the product of the hook lengths in the Young diagram of λ.

Thus for example we have $\dim S^{(3,2,1)} = \frac{6!}{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1} = 16$.

Many proofs of this result have been given since the original one in 1954 by Frame, Robinson, and Thrall, which used the representation theory of S_n in prime characteristic. A straightforward proof by induction is given in Fulton's book, together with a heuristic argument which can be converted to a rigorous probabilistic proof. More recently Pak and others have given a geometric proof, by computing the n-dimensional volume of a certain polytope.

It turns out that the representations of S_n as n varies fit together in a beautiful way. To see how this works, let R_n be the free $\mathbb{Z}$-module spanned by the irreducible characters of S_n. Make the direct sum $R = \bigoplus_{n=0}^{\infty} R_n$ into a graded ring by decreeing that the product $\chi \circ \mu$ of the irreducible characters χ, μ of S_m, S_n, respectively, be the induced character $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi \times \mu$ of S_{m+n}, where $\chi \times \mu$ is regarded as a character of the direct product $S_m \times S_n$ via $\chi \times \mu(g, h) = \chi(g)\mu(h)$ and $S_m \times S_n$ is embedded in S_{m+n} in the obvious way. A straightforward argument using the transitivity of induction (Proposition 14 in DF, p. 898) shows that multiplication in R is associative and commutative.

The ring R then turns out to be isomorphic to a well-known ring Λ arising in a different context. Recall that a polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ is called symmetric if it remains unchanged when the variables are permuted: $p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$. Note that p is symmetric if and only if the sum h_i of its monomial terms of degree i is symmetric for all i. A symmetric function $p = (p_1, p_2, \ldots)$ is then a tuple of symmetric polynomials p_i in $x_1, \ldots, x_i$, such that $p_j(x_1, \ldots, x_i, 0 \ldots, 0) = p_i(x_1, \ldots, x_i)$ for all indices i, j with $i < j$. For example, $p = (p_1, p_2, \ldots)$ has this property if we set $p_i = x_1 + \ldots + x_i$. It is clear that the sum and product of symmetric functions are symmetric. Thus $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$ has the structure of a commutative graded ring, where Λ_n denotes the set of homogeneous symmetric functions of degree n.

To any partition λ one can attach a symmetric function, as follows. We first broaden the definition of Young tableau, now declaring any filling of the boxes in a Young diagram by positive integers to be such a tableau. We say that the tableau T is semistandard if the numbers in the boxes increase weakly across the rows but strictly down the columns. For every semistandard tableau T denote by x^T (the *weight* of T) the monomial $x_1^{a_1} \dots x_m^{a_m}$, where the exponent a_i counts the number of times i appears in T. Then the Schur polynomial $s_{\lambda,m}$ in m variables is defined to be the sum $\sum_T x^T$ as T runs over the semistandard tableaux using only the numbers 1 through m (but not necessarily all of them).

For example, if λ has a single part n, then $s_{\lambda,m}$ is the nth complete symmetric polynomial $h_n(x_1, \ldots, x_m)$, equal to the sum of all monomials in $x_1, \ldots, x_m$ of total degree n. If instead $\lambda = (1, \ldots, 1)$ with n ones, then $s_{\lambda,m}$ equals $e_n(x_1, \ldots, x_m)$, the nth elementary symmetric polynomial in $x_1, \ldots, x_m$, equal to the sum of all products of n distinct variables among the x_i; note that $e_n(x_1, \ldots, x_m) = 0$ if $m < n$.

We further have $s_{(2,1),1} = 0$, since there are no semistandard tableaux of shape (2, 1) using only the number 1, while $s_{(2,1),2} = x_1^2 x_2 + x_1 x_2^2$. The coefficient of $x_1 x_2 x_3$ in $s_{(2,1),3}$ is 2, since there are two standard tableaux of shape (2, 1). As indicated by (but certainly not obvious from) the examples so far, it turns out that $s_{\lambda,i}$ is symmetric for every λ and i, whence $s_\lambda = (s_{\lambda,1}, s_{\lambda,2}, \ldots)$ is a symmetric function, called the Schur function s_λ corresponding to λ.

The connection between characters of S_n and symmetric functions is then established by the following fundamental result.

Theorem: Fulton, Chapter 7

The graded rings R and Λ are isomorphic by the map sending the character χ_λ of the Specht module S^λ (lying in R_n) to $s_\lambda \in \Lambda_n$.

This result is particularly remarkable since most of the tableaux T involved in the definition of s_λ are not standard and play no role in the definition of S^λ. The theorem mostly has to do with the representation theory of the infinite group $GL_n(\mathbb{C})$, which involves semistandard tableaux in a crucial way. The connection to S_n amounts to a side benefit.

By working with semistandard tableaux one can deduce the following restriction formula for representations of S_n.

Theorem: restriction formula

Given a partition $\lambda$of n, the restriction of the Specht module S^λ to S_{n-1} as the sum $\sum S^{\lambda'}$ as λ' runs through the partitions whose diagrams are obtained from that of λ by deleting one box. In particular, this restriction is irreducible if and only if the diagram of λ is a rectangle (all rows have the same length).

If one restricts S^λ to S_{n-1}, then to S_{n-2}, and so on, all the way to S_1, one obtains the sum of $f^\lambda = \dim S^\lambda$ copies of the trivial representation. So f^λ equals the number of ways to reduce the diagram D_λ of λ to a single box by removing one box at a time, making sure that resulting shape is always a diagram. Now the boxes that can be removed from D_λ at the first step are exactly those which can be filled by the largest number n in a standard tableau of shape λ, and similarly for the boxes that can be removed at the subsequent steps.

In this way we recover the earlier result that $\dim S^\lambda$ equals the number of standard tableaux of shape λ. We conclude with the induction analogue of the restriction formula: the representation induced from S_n to S_{n+1} by S^λ is the sum $\sum S^{\lambda''}$ as λ'' runs over the partitions whose diagrams are obtained from D_λ by adding one box, possibly forming a new row by itself.