Lecture 4-17: Specht modules and standard tableaux

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Given a partition λ of n, last time we defined the Specht module of S_n corresponding to λ to be the span of the $v_T = b_T \cdot \{T\}$ as T runs over all tableaux with shape λ, where $b_T = \sum_{q \in C(T)} \epsilon_q q$, where ϵ_q denotes the sign of q and $\{T\}$ the tabloid of T. We showed that $b_T v_T \neq 0$ and that $b_T\{T'\} \in \mathbb{Q} v_T$ for any tabloid $\{T'\}$ if T' has shape λ. Hence $b_T M^\lambda = b_T S^\lambda = \mathbb{Q} v_T \neq 0$, while $b_T M^{\lambda'} = 0$ if $\lambda' > \lambda$.

Theorem

The Specht modules S^λ are irreducible, as are their complexifications $S^\lambda_{\mathbb{C}}$ (obtained by replacing the basefield $\mathbb{Q}$ by $\mathbb{C}$). Every irreducible complex representation of S_n is isomorphic to $S^\lambda_{\mathbb{C}}$ for a unique partition λ of n.

Proof.

Irreducibility over either $\mathbb{C}$ or $\mathbb{Q}$ is equivalent to indecomposability. If we had $S^\lambda = V \oplus W$, then $\mathbb{Q}v_T = b_T S^\lambda = b_T \cdot V \oplus b_T \cdot W$, forcing one of V or W, say V, to contain v_T, whence $V = \mathbb{Q}S_n v_T = S^\lambda$ and S^λ is irreducible; similarly so is $S^\lambda_{\mathbb{C}}$. Since $<$ is a total order on tableaux, no two S^λ are equivalent, by the above formulas for $b_T \cdot M^\lambda$. Since the number of S^λ matches the number of conjugacy classes in S_n, we have found all of the irreducible complex representations of S_n. □

From this it can be shown that the rational group algebra $\mathbb{Q}S_n$ is isomorphic to a sum of matrix rings $M_{n_i}(\mathbb{Q})$, in the same way that $\mathbb{C}S_n$ is a sum of matrix rings $M_{n_i}(\mathbb{C})$.

Now we want to work out the degree of S^λ. We first show that this degree is at least the number f^λ of standard tableaux of shape λ.

Proposition

For fixed λ the elements v_T are linearly independent as T runs through the standard tableaux of shape λ.

Proof.

We mention that the argument in Chapter 7 of Fulton is inadequate, using only as it does the previously defined total order on tableaux. Instead one needs a total order $\prec$ on *tabloids* $\{T\}, \{T'\}$ of shape λ, defined by decreeing that $\{T\} \prec \{T'\}$ if the largest number occurring in different rows occurs higher in $\{T\}$ than in $\{T'\}$. By the remark after Lemma 1 of the last lecture, we have $\{q \cdot T\} \prec \{T\}$ if T is standard and $q \in C(T)$. Thus for T standard each v_T is a combination of tabloids of which the $\prec$-largest term is $\{T\}$. The independence of such v_T then follows at once by considering the $\prec$-maximal term occurring with nonzero coefficient in a dependence relation. □

Now we want to show that the v_T for T standard also span S^λ. To do this we need a different presentation of S^λ, realizing it as a quotient rather than a submodule of an S_n-module. We therefore define a column tabloid $[T]$ to be an equivalence class of tableaux $\pm T$ with signs attached, where $\pm T$ is identified with $\pm(\epsilon_q T)$ whenever $q \in C(T)$. There is an obvious action of S_n on column tabloids of shape λ and thus on the space $\tilde{M}^\lambda$ spanned by them; the definitions of $[T]$ and v_T show that the map $\alpha : \tilde{M}^\lambda \to S^\lambda$ sending $[T]$ to v_T is a well defined surjective S_n-module map.

It turns out that there are certain operations $\pi_{j,k}$ on column tabloids which are such that the differences $[T] - \pi_{j,k}[T]$ span the kernel of α. Here the parameter j is a column of T lying to the left of its rightmost column and k is a positive integer at most equal to the length of the $(j+1)$st column of T. Then $\pi_{j,k}[T]$ is the sum of the column tabloids $[S]$ obtained from $[T]$ by interchanging the top k numbers in the $(j+1)$st column with all possible subsets of k numbers in the jth column, preserving the vertical order of these elements throughout.

For example,

$$\pi_{1,2}\begin{pmatrix}1 & 2\\ 4 & 3\\ 5 & 6\end{pmatrix} = \begin{pmatrix}2 & 1\\ 3 & 4\\ 5 & 6\end{pmatrix} + \begin{pmatrix}2 & 1\\ 4 & 5\\ 3 & 6\end{pmatrix} + \begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}$$

For the proof that the differences $[T] - \pi_{j,k}[T]$ span the kernel of α, see Chapter 7 of Fulton's book; it is not difficult but a bit too much of a digression to include here. We now introduce one more total ordering, this time on column tabloids (the last one, I promise!) We decree that $[T] \succ [T']$ if in the rightmost column where $[T], [T']$ differ, the lowest box which has different entries after both columns are rearranged in increasing order is larger in $[T]$ than in $[T']$. Then we have

Lemma 3: straightening law

If S is a nonstandard tableau of shape λ then by repeatedly using the relations $[T] = \pi_{j,k}[T]$ (for various j, k) we can write $[S]$ as a combination of $[T_i]$ where the T_i are standard.

Proof.

First, we may rearrange the columns of S are in increasing order, possibly changing $[S]$ by a sign. If S is still not standard, then suppose that the kth number in the jth column is larger than the kth number in the $(j+1)$st column. Applying $\pi_{j,k}$ to S, we find that all column tabloids appearing are larger than $[S]$ in the ordering, so iteration of this process equates $[S]$ to a combination of the desired form. □

For example, taking $j = k = 1$, we find that

$$\begin{pmatrix} 2 & 1 \\ 3 & \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & \end{pmatrix}$$

Hence the v_T for T standard provide a basis of the Specht module S^λ and its dimension equals the number f^λ of standard tableaux of shape λ. Also we see that the Specht module S^λ can in fact be defined over the integers, so that every representation π of S_n is equivalent to one whose range lies in $GL_n(\mathbb{Z})$. In particular, we recover the result you proved in HW that the character table of S_n consists entirely of integers.

There is a similar dual construction of a module $\tilde{S}^\lambda$ isomorphic to S^λ, obtained by taking the span of the elements $\tilde{v}_T = a_T \cdot [T] \in \tilde{M}^\lambda$, where $a_T = \sum_{\sigma \in R(T)} \sigma$. Thus we have composite maps $S^\lambda \hookrightarrow M^\lambda \to \tilde{S}^\lambda$ and $\tilde{S}^\lambda \hookrightarrow \tilde{M}^\lambda \to S^\lambda$. The composite of these composites sending S^λ to itself is multiplication by a scalar, by Schur's Lemma. It maps v_T to $b_T a_T \cdot v_T = n_T v_T$, where n_T equals the cardinality of the set of quadruples (p_1, q_1, p_2, q_2) such that $p_i \in R(T), q_i \in C(T), p_1 q_1 p_2 q_2 = 1$ and $\epsilon_{q_1} = \epsilon_{q_2}$, minus the cardinality of the set of quadruples (p_1, q_1, p_2, q_2) satisfying the first two conditions but with $\epsilon_{q_1} = -\epsilon_{q_2}$. (In Fulton's book, the subtracted term is erroneously omitted.) This is independent of the choice of T (with shape λ) since replacing T by a different tableau replaces the subgroups $R(T), C(T)$ by conjugates of themselves. Taking T minimal in the order on tableaux of shape λ, we see that multiplication by c_T on $\mathbb{Q}S_n$ sends all c_U to 0 for all standard $U \neq T$ (even those not of shape λ), but acts with trace $n!$ on $\mathbb{Q}S_n$, so finally $n_\lambda = \frac{n!}{\dim S^\lambda} \neq 0$.

On the other hand, the representations S^λ and $\tilde{S}^\lambda$ are not in general equivalent over the integers, and the Specht modules S^λ, while still defined over any field k, need not be irreducible (or inequivalent) in general.