Lecture 4-15: Representations of the symmetric group

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We turn now to a particular group whose representations turn out to be particularly nice (and were historically among the first ones studied). This is the symmetric group S_n. We will follow the development in Chapter 7 of Fulton's wonderful 1997 book entitled "Young tableaux"; throughout we will work over the rational field $\mathbb{Q}$ rather than the complex one. First recall that the conjugacy class of a permutation σ is determined by the lengths of the cycles in its cycle decomposition (including the 1-cycles). These form a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n, so that $\sum \lambda_i = n$; by convention we arrange the parts λ_i of λ so that $\lambda_1 \geq \lambda_2 \geq \ldots$. We also write $|\lambda| = n$.

Given a partition λ of n, we define a (Young) diagram of shape λ to be a an arrangement of boxes in rows, lined up on the left, so that the ith row of the arrangement has λ_i boxes. Filling in the boxes with the numbers 1 through n, using each number exactly once, we get a (Young) tableau of this shape, which is called standard if the numbers in the boxes increase across rows and down columns.

Thus for example

$$\begin{matrix} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & & \end{matrix}$$

is a standard tableau of shape $(3, 2, 1)$. There is an obvious action of S_n on tableaux of shape λ, obtained by permuting the numbers in the boxes. Given such a tableau T, denote by $R(T)$ the subgroup of S_n consisting of permutations permuting the elements of each row among themselves. Then $R(T)$ is a direct product of symmetric groups, one for each part of λ.

Similarly denote by $C(T)$ the subgroup of permutations preserving the columns of T. In the above example $R(T)$ and $C(T)$ are both isomorphic to $S_3 \times S_2 \times S_1$. Note that $R(T) \cap C(T) = 1$, since a permutation in the intersection cannot move any number from its row or column in T. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ of the same integer n, we say that λ *dominates* λ' if for all i we have $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \lambda'_j$, defining $\lambda_j = 0$ if $j > m$ and $\lambda'_k = 0$ if $k > r$. This is a partial order on partitions of n. The following lemma provides the basic tool we need.

Lemma 1

Let T, T' be tableaux of shapes λ, λ' with $|\lambda| = |\lambda'| = n$. Assume that λ does not strictly dominate λ'. Then exactly one of the following holds.

- There are two distinct integers in the same row of T' and the same column of T.
- $\lambda' = \lambda$ and there are $p' \in R(T'), q \in C(T)$ with $p' \cdot T' = q \cdot T$.

Proof.

If the first assertion fails, then the numbers in the first row of T' all occur in different columns of T, so there is $q_1 \in C(T)$ such that these numbers occur in the first row of $q_1 \cdot T$. The numbers in the second row of T' then occur in different columns of T, so also of $q_1 \cdot T$, so there is $q_2 \in C(q_1 \cdot T) = C(T)$ not moving the numbers equal to those in the first row of T', such that these numbers all occur in the first two rows of $q_2 q_1 \cdot T$. Continuing in this way we get $q_1, \ldots, q_k \in C(T)$ such that the numbers in the first k rows of T' all occur in the first k rows of $q_k \cdots q_1 \cdot T$. Since T and $q_k \ldots q_1 \cdot T$ have shape λ, the sum of the first k parts of λ' can be at most the corresponding sum for λ and λ dominates λ'. □

Proof.

Since we have assumed the λ does not strictly dominate λ', we must have $\lambda = \lambda'$; taking k to be the number of rows of λ and $q = q_k \cdots q_1$, we see that $q \cdot T$ and T' have the same numbers in each row, so there is $p' \in R(T')$ with $p' \cdot T' = q \cdot T$, as desired; conversely, if such p', q exist, then the first assertion must fail. □

We now define two total orders, one on partitions and the other on tableaux. Given two distinct partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ we say that $\lambda > \lambda'$ (in the lexicographic order) if we have $\lambda_i > \lambda'_i$, where i is the smallest index for which $\lambda_i \neq \lambda'_i$. Given tableaux T, T' of respective shapes λ, λ' we write $T > T'$ if either $\lambda > \lambda'$ in the lexicographic order, or $\lambda = \lambda'$ and the largest number occurring in a different position in T and T' occurs either in a column further to the left in T or in the same column but lower down. Then for T standard, if $p \in R(T), q \in C(T)$, then $p \cdot T \geq T, q \cdot T \leq T$; indeed, the largest number in T moved by p is must be moved to the left, while the largest number moved by q must be moved up.

It follows that if T, T' are standard tableaux with $T' > T$ then there is a pair of numbers in the same row of T' and the same column of T. For then we must be in the second case of Lemma 1, so that $p' \cdot T' = q \cdot T$ for some p', q; but this forces $q \cdot T \leq T, p' \cdot T' \geq T'$ by the above observation, a contradiction.

We now define a tabloid $\{T\}$ to be an equivalence class of tableaux, two tableaux being equivalent if they have the same shape and the same numbers in each row. Thus the tableaux represented by

$$\begin{matrix} 1 & 4 & 7 \\ 3 & 6 & \\ 2 & 5 & \end{matrix}$$

and

$$\begin{matrix} 4 & 7 & 1 \\ 6 & 3 & \\ 2 & 5 & \end{matrix}$$

are the same. Clearly $\{T\} = \{T'\}$ if and only if $T' = p \cdot T$ for some $p \in R(T)$.

S_n acts on tabloids by the recipe $\sigma \cdot \{T\} = \{\sigma \cdot T\}$; thus the space M^λ spanned by all tabloids of shape λ is an S_n-module. For a tableau T, define $v_T = \sum_{\sigma \in C(T)} \epsilon_\sigma \sigma\{T\} = b_T\{T\}$, where $b_T = \sum_{\sigma \in C(T)} \epsilon_\sigma \sigma \in \mathbb{Q}S_n$, the rational group algebra of S_n, where ϵ_σ is the sign of σ (1 if σ is an even permutation, -1 otherwise). Clearly $v_T \neq 0$, since $R(T) \cap C(T) = 1$, whence $b_T v_T = b_T^2\{T\} = \#C(T) v_T \neq 0$, where $\#C(T)$ denotes the cardinality of $C(T)$. We have $\sigma \cdot v_T = v_{\sigma . T}$ for $\sigma \in S_n$ and all tableaux T. Now finally we define the Specht module S^λ to be the $\mathbb{Q}S_n$-module spanned by the v_T as T runs through tableaux of shape λ.

Irreducibility of the S^λ will follow from the following lemma.

Lemma 2

Let T, T' be tableaux of respective shapes λ, λ' and assume that λ does not dominate λ'. If there is a pair of integers in the same row of T' and column of T, then $b_T \cdot \{T'\} = 0$. Otherwise we have $b_T \cdot \{T'\} = \pm v_T$.

Proof.

If there is such a pair of integers, let t be the transposition that swaps them. Then $b_T t = -b_T$, since $t \in C(T)$, but $t \cdot \{T'\} = \{T'\}$, since $t \in R(T')$. It follows that $b_T \cdot \{T'\} = -b_T \cdot \{T'\} = 0$. If there is no such pair, choose p' and q as in the second case of Lemma 1. Then

$b_T \cdot \{T'\} = b_T \cdot \{p' \cdot T'\} = b_T \cdot \{q \cdot T'\} = b_T \cdot q \cdot \{T\} = \epsilon_q b_T \cdot \{T\} = \epsilon_q \cdot v_T$. □

By the remark right after Lemma 1, we deduce that if T, T' are standard tableaux with $T' > T$ then $b_T \cdot \{T'\} = 0$.