Lecture 4-12: Induced characters and representations

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Given a representation V of a group G, one can clearly restrict to a subgroup H, whose character is just the restriction χ_H to H of the character χ of V. There is an operation called *induction* going the other way, from characters of H to characters of G, which provides a very useful way of obtaining representations of larger groups from those of smaller ones.

Given a class function c_H on a subgroup H of a group G, how could one use it to produce a class function c_G on G? To make sure that c_G really is a class function, one needs to average over conjugacy classes in G, but the original function c_H is defined only on H. One is fairly naturally led to

Definition: DF p. 849, Corollary 12

We define $c_G = \mathrm{Ind}_H^G c_H$, the function induced from c_H, via $c_G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} c_H(x^{-1}gx)$, where $|H|$ denotes the order of H.

Equivalently, letting $g_1, \ldots, g_m$ to be a set of representatives of the left cosets of H in G, we take $c_G(g) = \sum_{g_i^{-1}xg_i \in H} c_H(g_i^{-1}xg_i)$.

Clearly $c_G = \mathrm{Ind}_H^G c_H$ depends linearly on c_H.

If c_H is the character χ of a representation of H, then c_G is also the character of a representation. This follows from the next result, in which $(\cdot, \cdot)$ denotes the standard Hermitian inner product on characters defined earlier.

Theorem: Frobenius reciprocity

With notation as above, we have $(c_G, \rho) = (c_H, \rho_H)$ for all characters ρ of G.

Proof.

We have

$$(c_G,\rho) = \frac{1}{|G|}\sum_{g\in G} c_G(g)\overline{\rho(g)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{\substack{x\in G\\ x^{-1}gx\in H}} c_H(x^{-1}gx)\overline{\rho(x^{-1}gx)}.$$

In this last sum every term $c_H(h)\overline{\rho(h)}$ appears exactly $|G|$ times for each fixed $h \in H$, whence the sum equals $(c_H, \rho_H) = (\rho_H, c_H)$, as claimed. □

We express this last result by saying that induction from H to G is the left adjoint of restriction from G to H: we move from the left to the right side of the inner product by replacing an induced representation by a restricted one.

If c_H is the character of an irreducible representation V of H, then every irreducible representation W of G decomposes as a direct sum of irreducible representations of H, including V some nonnegative number n_W of times, whence c_G is the character of the direct sum of n_W copies of W as W runs over the irreducible representations of G. If c_H is the character of an arbitrary representation of H then by linearity c_G is again the character of a representation.

It is also possible to construct the induced representations directly, as follows (DF, p. 893). Given a representation V of H, form the tensor product $W = \mathbb{C}G \otimes_{\mathbb{C}H} V$, regarding $\mathbb{C}G$ as a left module over itself and a right module over $\mathbb{C}H$, so that $xy \otimes v = x \otimes yv$ in W if $x \in \mathbb{C}G, y \in \mathbb{C}H, v \in V$, and $\mathbb{C}G$ acts on W by left multiplication on the left factor $\mathbb{C}G$. Then the degree of W is $\frac{|G|}{|H|}$ times the degree of V; more generally, any class function c_G on G induced by a class function c_H on H is such that $c_G(1) = \frac{|G|}{|H|} c_H(1)$.

As an example, take $G = S_3$ and let H be the cyclic subgroup generated by a transposition. Inducing the trivial character from H to G, we get a character taking the value 3 at the identity e, 1 on a transposition, and 0 on a 3-cycle; taking the square length of this character, we find that it is the sum of two irreducible characters. Subtracting off the trivial character (which occurs in it by Frobenius reciprocity) we get the character denoted by χ_r in a previous lecture, taking the value 2 at e, 0 on a transposition, and -1 on a 3-cycle.

Thus if we had never heard of the representation with this character, we could reconstruct it by induction. A similar calculation shows that inducing the nontrivial character from H to G gives the character with value 3 at e, -1 on a transposition, and 0 on a 3-cycle; subtracting off the character χ_r just constructed, we recover the character of the sign representation, which is 1 on e and a 3-cycle and -1 on a transposition.

We can also get interesting information by inducing class functions that are not characters. As an example, start with the cyclic subgroup T generated by a 3-cycle in the alternating group A_4 on four letters. Take each of the one-dimensional characters of T, subtract off the trivial character, and induce the resulting class function to A_4. You get $\chi_1 - \chi_1, \chi_2 - \chi_1, \chi_3 - \chi_1$, where the χ_i range over the three one-dimensional characters of A_4 (constructed previously; here χ_1 is the trivial character). What is going on in both of these examples is that there is a subgroup C of G, equal to H in the first example and T in the second, which is such that $g^{-1}Cg \cap C = 1$ for any $g \in G$ with $g \notin C$.

More generally, let G be a transitive subgroup of the symmetric group S_n (acting on the index set $\{1, \ldots, n\}$ with a single orbit) such that only the identity in G fixes as many as two indices among $1, \ldots, n$. Let H be the stabilizer in G of any index, say of 1. The hypothesis on G implies that $H \cap g^{-1}Hg = 1$ whenever $g \notin H$. A famous result of Frobenius then asserts that the set of permutations in G fixing no index, together with the identity, is a normal subgroup of G; this is called the *Frobenius kernel* while the subgroup H is called the *Frobenius complement*. You will prove this in homework next week, using the theory of induced characters.

As a final example, it is not difficult to check that if you induce the character of either V_{n-1} or V'_{n-1} of the Clifford group G_{n-1} for n even, you get the character of V_n on G_n.