Lecture 4-10: Clifford algebras

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We digress to study the representations of a particular family of groups, which I privately call the Clifford groups, in detail. This will lead to an important class of algebras universally called (complex) Clifford algebras, which are of considerable independent interest and importance. We will also study real Clifford algebras, which exhibit an even richer structure.

We begin with the group G_n. It is generated by n elements $a_1, \ldots, a_n$, with the defining relations $a_i^2 = \epsilon, \epsilon^2 = 1$ and $a_i a_j = \epsilon a_j a_i$ for $i \neq j$. Note that if $n = 2$ this is just the group of quaternion units, generated by the quaternions i and j, with $\epsilon = -1$. In general there does not seem to be any standard name for this group in the literature; the term "Clifford" group is probably as good as any other. The center Z_n of G_n has order 2 and is generated by ϵ if n is even; it is generated by ϵ and $z_n = a_1 \ldots a_n$ if n is odd. More precisely, Z_n is cyclic of order 4 if $n \equiv 1 \mod 4$; it is isomorphic to the Klein four-group if $n \equiv 3 \mod 4$. For any $g \in G_n$, the conjugacy class of g consists of g alone if $g \in Z_n$ and of g and $g\epsilon$ if $g \notin Z_n$. Thus G_n has $2^n + 1$ conjugacy classes if n is even and $2^n + 2$ conjugacy classes if n is odd.

We first dispose of the boring representations of G_n; that is, those of dimension 1. These are the ones for which ϵ acts trivially. The quotient $G_n/\langle\epsilon\rangle$ of G_n by the subgroup generated by ϵ is the direct product of n copies of the cyclic group $\mathbb{Z}_2$; accordingly G has 2^n one-dimensional representations.

There is room for only one more irreducible representation of G_n if n is even, necessarily of degree $2^{\frac{n}{2}}$, since the sum of the squares of the irreducible degrees is then $2^n + 2^n = 2^{n+1}$. the order of G_n. If n is odd, there are two more irreducible representations; since each has degree a power of 2 and the sum of the squares of the degrees must again be 2^{n+1}, both must have degree $2^{\frac{n-1}{2}}$. In all of these cases $\epsilon \in G_n$ acts by a scalar square root of 1, so it must act by -1.

The complex Clifford algebra C_n is then defined to be the quotient of the group algebra $\mathbb{C}G_n$ by the ideal generated by the central element $\epsilon + 1$. This is a single matrix algebra $M_{\frac{n}{2}}(\mathbb{C})$ if n is even and the direct sum $M_{\frac{n-1}{2}}(\mathbb{C}) \oplus M_{\frac{n-1}{2}}(\mathbb{C})$ if n is odd. It can also be thought of the algebra over $\mathbb{C}$ generated by $a_1, \ldots, a_n$ with the defining relations $a_i^2 = -1, a_i a_j = -a_j a_i$ for $i \neq j$. Sometimes one takes a_i^2 to be 1 rather than -1; this leads to an isomorphic algebra.

If n is odd, there is a very simple relationship between the two irreducible representations of degree $2^{\frac{n-1}{2}}$. If $n \equiv 3 \mod 4$, then G_n is isomorphic to the direct product of G_{n-1} and the cyclic subgroup $\langle z_n \rangle$ generated by z_n, which has order 2. Starting from the unique irreducible representation V_{n-1} of G_{n-1} of degree larger than one, we extend it to G_n by having z_n act either trivially or by -1, thus obtaining the two representations V_n, V_n' of G_n of degree larger than one. If instead $n \equiv 1 \mod 4$, then $z_n^2 = \epsilon$, so z_n has order 4. Starting with V_{n-1} as before, extend it to G_n by having z_n act by either $\pm\sqrt{-1} \in \mathbb{C}$, once again obtaining V_n and V_n'.

If n is even, then the representation V_n turns out to decompose over G_{n-1} as the sum of the two representations V_{n-1}, V'_{n-1}. We will see later that any representation of a subgroup H of a group G can be "induced" to a larger representation of G; inducing either V_{n-1} or V'_{n-1} from G_{n-1} to G_n, one realizes the representation V_n. The character table of G_n is easy to compute, since any non-central $g \in G_n$ is conjugate to ϵg, which acts by the negative of the action of g on any irreducible representation of degree larger than one, whence its character on that representation is 0.

We get even more interesting behavior by replacing the basefield $\mathbb{C}$ with $\mathbb{R}$. In general, the *real* group algebra $\mathbb{R}G$ of a finite group G is not a direct sum of matrix rings over $\mathbb{R}$; instead it is a sum of matrix rings over (any or all) of $\mathbb{R}, \mathbb{C}$, and the quaternions $\mathbb{H}$. For example, defining the real Clifford algebra R_n by the same generators and relations as above, we see that R_2 satisfies the same relations as $\mathbb{H}$, so is isomorphic to it. In general R_n is either a single matrix algebra or the sum of two isomorphic ones. Its structure (that is, whether there are one or two matrix algebras and whether they are real, complex, or quaternionic) is periodic with period eight; all of this is related to a topological phenomenon called *Bott periodicity*.

The real Clifford algebra C_n is not a group under multiplication, but it has a large subset which is a group, isomorphic to the simply connected double cover $\mathrm{Spin}(n, \mathbb{R})$ of the special orthogonal group $SO(n, \mathbb{R})$. Similarly, the complex Clifford algebra C_n has a copy of the double cover $\mathrm{Spin}(n, \mathbb{C})$ of $SO(n, \mathbb{C})$ sitting inside it.

Finally, I mention the *composition problem*, which asks for which positive integers n there exist real numbers b_{ijk} for $1 \le i, j, k \le n$ such that for all $x_1, \ldots, x_n, y_1, \ldots y_n \in \mathbb{R}$ such that if we set $z_i = \sum_{j,k} b_{ijk} x j y_k$ then we have an identity $(\sum_i x_i^2)(\sum_i y_i^2) = \sum_i z_i^2$. It turns out that such an identity exists for exactly four values of n, namely 1,2,4, and 8. The identity for $n = 1$ is trivial. For $n = 2$ it comes to the multiplicativity of the complex norm; likewise for $n = 4$ it comes down to the multiplicativity of a norm, this time the quaternionic norm. For $n = 8$ there is another set of numbers called the *octonions* admitting a multiplicative real-valued norm. The octonions are even worse behaved than the quaternions: in addition to being noncommutative they are not even associative under multiplication. This roughly explains why one does not just iterate the procedure to get the quaternions from the complex numbers ad infinitum.

That the composition problem has no solution for any n other than $1, 2, 4$, or 8 comes down to a calculation using the representation theory of R_n.