Lecture 3-27: Jordan decompositions

March 27, 2024

Last time we saw the Jordan canonical form of a square matrix M over a field k containing all eigenvalues of M (in any extension field), which realizes M as similar to a block diagonal matrix for which the (Jordan) blocks B have equal entries along their main diagonals, 1s just above these diagonals, and 0s elsewhere. In particular, we can write each B as the sum of the diagonal matrix B_D having the same diagonal entries as B and a strictly upper triangular matrix B_N, whose nonzero entries are the 1s occurring in B.

It is clear for each B that the matrices B_D and B_N commute with each other. Putting the blocks together and passing to similar matrices, we deduce that under the above hypothesis on k, any square matrix is similar to a sum $D+N$ of a diagonalizable matrix D and a nilpotent matrix N where D and N commute. Thus N such that $N^m = 0$ for some positive integer m. But now it turns out that we can say even more about the matrices D and N in this sum. To do this, we start over, writing the ambient vector space V as the direct sum $\bigoplus_{i=1}^{m} V_i$ of subspaces V_i preserved by M, such that for each i there is a scalar $\lambda_i \in k$ with $(M - \lambda_i I)^{n_i}$ acting by 0 for some integer n_i, with the λ_i distinct.

Now use the Chinese Remainder Theorem to find a polynomial $p \in k[x]$ such that $p(x) \equiv \lambda_i \mod (x - \lambda_i)^{n_i}$ for each i while $p(x) \equiv 0 \mod x$; note that we can omit the last congruence if some $\lambda_i = 0$, so that the moduli are always pairwise relatively prime. Then the polynomial $p(M)$ acts as the scalar λ_i on each V_i and p has constant term 0; moreover, $q(M) = M - p(M)$ acts nilpotently on each V_i (and thus on all of V), with the constant term of q also being 0. The upshot is that the matrices $D = p(M), N = q(M)$ are diagonalizable and nilpotent, respectively. Both commute with M (being polynomials in M) and they commute with each other

Moreover, $D + N$ is the *only* decomposition of M as the sum of commuting diagonalizable and nilpotent matrices. Indeed, given such a decomposition $D' + N'$, we have $D - D' = N - N'$ and D, D', N, N' commute pairwise, since all four matrices commute with M and D, N are polynomials in N. Now the binomial theorem (valid for any pair of commuting elements in any ring) shows that $N - N'$ is also nilpotent.

Likewise $D - D'$ is diagonalizable. To see this, decompose the vector space V as the sum of eigenspaces of D. Since D' commutes with D, it stabilizes all of its eigenspaces and must act diagonalizably on each one. Hence there is a basis of V which simultaneously diagonaiizes D and D'; this same basis realizes $D - D'$ as diagonalizable.

The upshot is that $D - D' = N - N'$ is both diagonalizable and nilpotent, whence this difference must be 0, as claimed. We have proved

Jordan decomposition theorem

Under the above hypothesis on k, any square matrix M is uniquely the sum $D + N$ of commuting diagonalizable and nilpotent matrices.

This basis-free version of the Jordan canonical form has many applications to Lie algebras, Lie groups, and algebraic groups. The summands D, N are called the semisimple and nilpotent parts of M. This terminology is explained below.

What about fields k not containing the eigenvalues of a square matrix M? It turns out that a weaker version of the Jordan decomposition (though not the Jordan canonical form) holds for most such fields. Call a square matrix D' *potentially diagonalizable* if it is similar to a diagonal matrix when regarded as a matrix over some extension k' of k. Then we have

Theorem

If k is perfect (so that it either has characteristic 0 or it has characteristic p and is closed under taking pth roots) then any square matrix M over k can be uniquely written as the sum $D' + N$ of commuting potentially diagonalizable and nilpotent matrices.

For the proof see the Wikipedia article "Jordan-Chevalley decomposition" (but be warned that the argument there is garbled and needs fixing).

We now define semisimple linear transformations. Note first that given an irreducible polynomial $p(x)$ over any field k, the quotient ring $k[x]/(p(x))$ has no proper ideals. Thus when regarded as a vector space V over k, it has no proper subspaces stable under multiplication by x. More generally, given a product $q = p_1(x) \ldots p_m(x)$of distinct irreducible polynomials $p_i(x)$, the quotient $k[x]/(q)$ is isomorphic to the direct sum of the quotients $V_i = k[x]/(p_i(x))$. Then multiplication by x acts on each V_i without stabilizing any proper subspace of it.

A linear transformation T on a finite-dimensional vector space V is called semisimple if V can be written as the direct sum of subspaces V_i stabilized by T such that T does not stabilize any proper subspace of V_i. For example, a nontrivial rotation not by π radians is semisimple as a linear transformation of $\mathbb{R}^2$. Then it turns out that over a perfect field a transformation is semisimple if and only if it is potentially diagonalizable. In particular, if k contains all eigenvalues of T, then T is semisimple if and only if it is diagonalizable. Any potentially diagonalizable linear transformation is semisimple, but the converse can fail for imperfect fields.

The notion of semisimplicity actually has many different meanings in algebra, but all involve finite direct sums. You learn more about some of the other meanings of this term later.