FINAL EXAM SOLUTIONS

1. Find two examples of a ring R with exactly two prime ideals, one with one such ideal contained in the other, the other with neither ideal contained in the other.

For the first example one can take any DVR, e.g. the power series ring $k[[x]]$, with k a field; for the second one can take e.g. $\mathbb{Z}/6\mathbb{Z}$.

2. Classify as completely as you can the fields that are homomorphic images of $\mathbb{Q}[x, y, z]$, the polynomial ring in three variables over $\mathbb{Q}$.

These are exactly the finite extensions of $\mathbb{Q}$ (any such extension being in fact generated by a single element over $\mathbb{Q}$).

3. Find all singular points on the affine curve in $\mathbb{C}^2$ defined by the equation $y^2 = x^3 + x^2$ and find the dimension of the tangent space to this curve at each such point.

There is only one such point, namely the origin $(0, 0)$, this being the only point at which the gradient ∇f of the defining equation $f(x, y) = y^2 - x^3 - x^2 = 0$ vanishes.

4. Let k be a field. Find a primary decomposition of the ideal $I = (xy, y^2)$ in the polynomial ring $k[x, y]$ in two generators over k; that is, realize I as a finite intersection of primary ideals. Also identify the radicals of each of these ideals.

We have $I = (y) \cap (x^2, y)$; the radicals of these primary ideals are (y) and (x, y).

5. If K is an algebraically closed field, classify the subvarieties of K^n whose coordinate rings are Artinian.

These are exactly the finite subsets of K^n.

6. The example of the curve C in $\mathbb{C}^2$ defined by the equation $x^3 = y^2$ has been discussed multiple times in class. State as many properties of this curve and its coordinate ring as you can, focussing on the ones that make it different from most curves and their coordinate rings.

The coordinate ring $\mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$ of this curve is not a Dedekind domain, since t lies in its quotient field but not in the ring itself. The variety C is not smooth, having the origin $(0, 0)$ as its unique singular point. We can regard C as an unramified double or unramified triple covering of $\mathbb{C}$, projecting to the first or second coordinate. C is birational but not isomorphic to $\mathbb{C}$; there is a bijective morphism from $\mathbb{C}$ to C whose inverse is not a morphism.

7. Show that the quotient field K of an integral domain R is flat as an R-module.

It was shown in class that the tensor product $M \otimes_R K$ of an R-module M with K may be identified with the localization $S^{-1}M$, with S the set of nonzero elements of R. The functor $M \to S^{-1}M$ is well known to be exact.

8. Let k be an algebraically closed field, A an affine domain over k. We have computed the dimension of A in four different ways (three of them involving a choice of maximal ideal M of A, but all three giving the same answer for any given M). Describe these ways as clearly as you can (but without giving proofs).

First of all, without choosing any M, we can take the dimension of A to be the transcendence degree of its quotient field over k. Next, having chosen M, we can take $\dim A$ to be the degree of the length $\ell(A/M^n)$ of A/M^n as a module over itself, regarded as an Artinian ring, this length being polynomial in n for sufficiently large n. We can also take $\dim A$ to be the maximum length d of a strict chain of prime ideals $P_0 \subset P_1 \cdots \subset P_{d-1}$, as the last element P_{d-1} of the chain runs over all maximal ideals of A. Finally, we can take $\dim A$ to be the minimum number of generators of any M-primary ideal of A, again as M runs over all maximal ideals of A.