Lecture 6-6: Review, part 3

June 6, 2025

I will wrap up the course with a few remarks on flatness and blowups.

Recall first that a module M over a ring R is called flat if tensoring with M is exact, so that it takes exact sequences of R-modules to exact sequences. Since tensoring is always right exact, M is flat if and only if tensoring with M is left exact, or equivalently takes injections to injections. Since free modules are easily seen to be flat and tensor products commute with direct sums, one sees at once that any projective module is flat, being a direct summand of a free module. Localizations of rings are likewise flat over the rings; these are typically not projective. Like injectivity, flatness can be verified by looking just at ideals: M is flat if and only the map from $I \otimes_R M \to M$ is injective for all ideals I of R.

The tensor product functor, like the Hom functor studied in the fall, admits higher derived functors, denoted $\mathrm{Tor}_i^R(M, N) = \mathrm{Tor}_i^R(N, M)$; thus $\mathrm{Tor}_0^R(M, N) = M \otimes N$ while given a short exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules and an R-module M, one gets a long exact sequence including $\mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^R(M, N'')$ for all $i > 0$ and ending with $M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$. One can compute $\mathrm{Tor}_i^R(M, N)$ by starting with a free (or more generally a projective) resolution of N, tensoring each term with M, and then taking the homology of the resulting chain complex.

Using this definition, it is easy to compute the Tor groups in some simple cases. For example, if $x \in R$ is a non-zero divisor and M is an R-module, then we have $\mathrm{Tor}_0^R(R/(x), M) = M/xM$, $\mathrm{Tor}_1^R(R/(x), M) = {}_xM = \{m \in M : xm = 0\}$, $\mathrm{Tor}_i^R(R/(x), M) = 0$ for $i \geq 2$. If R is a PID, then M is flat over R if and only if it is torsion-free. If $R = k[t]/(t^2)$ and M is an R-module, then M is flat if and only if multiplication by t from M to tM induces an isomorphism $M/tM \to tM$. Given local Noetherian ring with maximal ideal I and S a local Noetherian R-algebra with IS contained in its maximal ideal J, a finitely generated S-module M is flat over R if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$.

Finally, an important class of non-flat extensions of a ring R are its blowups $B_I R$ with respect to an ideal I, defined to be the direct sums $R \oplus I \oplus I^2 \oplus \cdots \cong R[tI] \subset R[t]$. On the level of algebraic sets over a field k, one starts with such a set X and an algebraic subset Y with vanishing ideal I, lets $a_1, \ldots, a_r$ be k-algebra generators of the coordinate ring R of X and $g_0, \ldots, g_s$ be ideal generators of I. The algebra $B_I R$ is the image of the algebra $k[x_1, \ldots, x_r, y_0, \ldots, y_s]$ under the map sending x_i to a_i and y_j to $g_j t$. The kernel of this map corresponds to an algebraic subset Z of $\mathbf{A}^r \times \mathbf{P}^s$; the first projection map maps Z onto X and is an isomorphism away from the preimage of Y. The preimage of Y in Z corresponds to the ring $B_I R / I B_I R = G_I R = \oplus_{n=0}^{\infty} I^n / I^{n+1}$. Roughly speaking, one replaces every point of Y with a copy of a suitable projective space.

In conclusion, give yourselves all a big pat on the back (particular the undergraduates): you have learned far more than I did in my first year of graduate school. Best of luck to all of you in your future work.