Lecture 6-2: Review, part I

June 2, 2025

I will spend the last week on review of the course material, beginning with algebraic geometry and moving on to commutative algebra today. As I did last quarter, I will concentrate on *statements* of theorems throughout; don't worry about memorizing their proofs, but be able to apply the theorems.

I begin with Noether normalization, which is the key tool used to prove the Nullstellensatz bijection between algebraic subsets of k^n and radical ideals in $k[x_1 \ldots, x_n]$. First let k be any field. Noether normalization asserts that any finitely generated k-algebra A is a finite integral extension of a polynomial ring $k[y_1, \ldots, y_d]$ over k; as a consequence, A is a field if and only if it is a finite extension of k. If k is algebraically closed, then the only such A that is a field is k itself. From this the Nullstellensatz asserts that for such k there is an order-reversing bijection between radical ideals of $k[x_1, \ldots, x_n]$ and algebraic subsets of k^n, mapping an ideal I to the set $Z(I)$ of its common zeros in k^n and a subset S to the ideal $I(S)$ of all polynomials vanishing at S.

An ideal I of $S = k[x_1, \ldots, x_n]$ is radical if and only if it is a finite intersection of prime ideals; on the geometric side, the zero set $Z(I)$ is uniquely a finite union of varieties, or irreducible algebraic sets, none of them contained in any of the others, so that $Z(I)$ is irreducible if and only if I is prime. More generally, any ideal I of P is a finite intersection $\cap P_i$ of primary ideals P_i, so that every zero divisor in S/P_i is nilpotent. The radical Q_i of each P_i is prime and the Q_i and are uniquely determined by I; also the P_i corresponding to ideals Q_i not contained in other Q_j are uniquely determined by I. More generally, these properties for any ideal I in a Noetherian ring R and any radical ideal of R is a finite intersection of prime ideals.

The most fundamental notion of size of a variety V, or more generally of an algebraic set A, is that of dimension; this is more general than but quite analogous to the dimension of a vector space. It was ultimately defined in three different ways in class. The most elementary definition looks at strictly increasing chains of subvarieties of V. If $V_0 \subset \cdots \subset V_{d-1}$ is a maximal such chain, then the dimension of V is taken to be d. More algebraically, the transcendence degree of the quotient field of the coordinate ring $k[V]$ is also equal to the dimension of V. In either case, the dimension of an algebraic set A is the largest dimension of any of its irreducible components. Algebra and geometry come together in yet another definition, namely that of the Krull dimension of $k[V]$. This is equal to the largest d such that there is a strictly increasing chain $P_0 \subset \cdots \subset P_{d-1}$ of prime ideals of $k[V]$; it too coincides with the dimension of V.

Yet another characterization (not quite a definition) of the dimension of V comes from Noether normalization: if $k[V]$ is a finite integral extension of a polynomial ring $k[y_1, \ldots, y_d]$, then d is the dimension of V. This characterization holds also for algebraic sets, without having to consider their irreducible components. Note that a 0-dimensional algebraic set A is just a finite union of points. Such sets are precisely the ones such that the coordinate ring $k[A]$ is Artinian.

In the more general setting of a commutative Noetherian ring R, one first localizes R at a maximal ideal M to compute its dimension. (I will review localization next time). Then the length $\ell(R/M^n)$ of the Artinian ring R/M^n as a module over itself is a polynomial in n for sufficiently large n; the degree d of this polynomial is the dimension of R. Alternatively, since the notion of Krull dimension is well defined for R, the can also be taken as its dimension; note that any maximal chain of prime ideals in R necessarily ends at the unique maximal ideal M. Finally, the minimum number of generators required for an M-primary ideal of R (with radical M) is also the dimension of R. In particular, the dimension of a Noetherian local ring is always finite. In general, the dimension of any Noetherian ring is defined to be the maximum possible dimension of any of its localizations R_M at a maximal ideal M; this can be infinite if R has infinitely many maximal ideals.

The notions of dimension of a variety on the one hand and a vector space on the other come together in the definition of the tangent space to a variety at a point, or more generally of a Noetherian ring R at a maximal ideal M; this is just the dimension of the quotient M/M^2 as an R/M-module. This is bounded below by the dimension of the localization R_M of R at M, but need not coincide with it in general. If the two coincide, R is said to be regular at M, or just regular if M is the only maximal ideal of R; if R is the coordinate ring $k[V]$ of a variety V, so that M corresponds to a point v of V, then such a v is called a smooth point of V. Any variety $V \subset k^n$ over an algebraically closed field k then has a nonempty open subset of smooth points; at all other points (called singular) the tangent space has larger dimension than it does at the smooth points. If the ideal $I(V)$ of V is generated by the polynomials $f_1, \ldots, f_r$, then the rank of the Jacobian matrix J of the f_i at a point $v \in V$ is generically equal to $n - d$, where d is the dimension of V.

An important example of a variety that was repeatedly mentioned in class is the curve C in k^2 defined by the equation $x^3 - y^2 = 0$. The coordinate ring of C is isomorphic to the subring $R = k[t^2, t^3]$ of $k[t]$ generated by t^2 and t^3. It fails to be a Dedekind domain, since $t = t^3/t^2$ lies in its quotient field and is integral over R but does not lie in R itself. We can regard C as either a ramified double or a ramified triple cover of the affine line k^1 via the projection maps to the first or second coordinate, bearing in mind that typical point on C takes the form (t^2, t^3) for a unique $t \in k$. The origin $(0, 0)$ is then the unique singular point of C, since the tangent space there is two-dimensional. Finally, C is birational but not isomorphic to k^1, via the map sending (t^2, t^3) to t for $t \neq 0$.

You have also seen some non-affine varieties, primarily the projective ones. Here projective space $\mathbf{P}^n$ is defined to be the set of lines L through the origin in k^{n+1}, or equivalently as the set of points x other than the origin, modulo the equivalence relation $x \sim ax$ for all $a \in k^*$. The homogeneous Nullstellensatz asserts the existence of an order-reversing bijection between homogeneous radical ideals of $k[x_0, \ldots, x_n]$ other than $(x_0, \ldots, x_n)$ and algebraic subsets of $\mathbf{P}^n$. The dimension of a subvariety V of $\mathbf{P}^n$ with corresponding ideal $I \subset k[x_0, \ldots, x_n]$ is one less than the dimension of the affine variety $C(V)$ corresponding to same ideal I. Any projective variety is a finite union of affine varieties, each lying in one of the affine pieces isomorphic to $\mathbf{A}^n$ that make up $\mathbf{P}^n$. Instead of a single coordinate ring $k[V]$ attached to V one has a sheaf of functions on the affine open subsets of V; the only regular functions defined on all of V are constants.

Finally, recall that in the Zariski topology on k^n (for k an algebraically closed field) the closed sets (by definition) are the zero sets $Z(I)$ of ideals I in the corresponding polynomial ring $S = k[x_1, \ldots, x_n]$. For $k = \mathbb{C}$ this topology is coarser than the usual Euclidean topology, in that there are many fewer open sets and the nonempty ones are much larger. In particular, every nonempty open subset of k^n is dense, as it is for any variety (or more generally for any irreducible topological space). Thanks to the Nullstellensatz, we can identify the points of k^n with the maximal ideals of S. More generally, for any commutative ring R (including even non-Noetherian ones) the (prime) spectrum of R consists of all the prime ideals of R. One defines a topology on this space by decreeing that the closed sets are the sets $V(I)$ of prime ideals containing a fixed ideal I.

In the special case where $R = k[A]$ is the coordinate ring of an algebraic subset A of k^n, one finds that the spectrum Spec R has a proper subset of points in bijection to the points of A; these correspond to the maximal ideals of R and are closed in the topology. Then there are additional points, corresponding to the non-maximal prime ideals of $k[A]$; these correspond to the subvarieties of A. The closure of the point corresponding to the subvariety V consists of the points corresponding to all subvarieties of V.