Lecture 5-7: Noetherian property of completions

May 7, 2025

I will wrap up Chapter 10 of Atiyah-Macdonald, showing that the completion $\hat{R}$ of a Noetherian ring R is again Noetherian.

Last time I proved Krull's Theorem, which asserts that the kernel of the natural map from a Noetherian ring R to its I-adic completion consists of the $x \in R$ annihilated by $1 - i$ for some $i \in I$. I now digress briefly to show that this result can fail if R is not Noetherian. Let $R = C^\infty(\mathbb{R})$, the ring of real-valued functions on $\mathbb{R}$ having derivatives of all orders, and let I be the principal ideal (x) of R. By L'Hopital's Rule, I consists exactly of the functions $f \in R$ with $f'(0) = 0$; similarly I^n consists of all $f \in R$ with $f^{(i)}(0) = 0$ for all $i \leq n$. The intersection $K = \cap_n I^n$ consists of all $f \in R$ with $f^{(n)}(0) = 0$ for all n.

Recall now from advanced calculus that $K \neq 0$: the function $g(x)$ defined to be e^{-1/x^2} if $x \neq 0$ and 0 if $x = 0$ lies in K. On the other hand, this function is not annihilated by any function in $1 + I$, for if $hg = 0$ with $h \in R$, then $h(x) = 0$ for $x \neq 0$, forcing $h = 0$ by continuity. Thus Krull's Theorem fails for R; in particular R is not Noetherian. By contrast, the ring of real-valued functions on $\mathbb{R}$ that are analytic at 0 *is* Noetherian, and in fact a DVR, its unique nonzero prime ideal being generated by the function x. The proof is the same as for the power series ring $\mathbb{R}[[x]]$. Thus rings of convergent power series behave algebraically much like rings of formal ones, but rings of smooth functions behave quite differently.

Returning now to a general ring R with ideal I, recall from the lecture on April 18 that the graded ring $G(R) = G_I(R)$ is defined to be $\oplus_{n=0}^{\infty} G_n = \oplus_{n=0}^{\infty} I^n/I^{n+1}$, with the product $\bar{x}_n\bar{x}_m$ defined for $\bar{x}_n \in G_n\bar{x}_m \in G_m$ by choosing preimages $x_n \in I^n, x_m \in I^m$ of $\bar{x}_n, \bar{x}_m$, respectively, and taking $\bar{x}_n\bar{x}_m$ to be the image of $x_n x_m$. Given an R-module M and an I-filtration (M_n) of M, I defined the graded module $G(M) = G_I(M)$ in a similar way to $G(R)$ and observed that $G(M)$ is a $G(R)$-module. I also showed that if R is Noetherian, M is finitely generated, and the filtration (M_n) is stable, then $G(M)$ is finitely generated over $G(R)$; moreover, the results of last time show that $G_I(R) \cong G_{\hat{I}}(\hat{R})$.

Now let A, B be filtered groups, so that we are given chains of subgroups $A_0 = A \supset A_1 \supset \cdots$ and $B = B_0 \supset B_1 \cdots$ of A and B, respectively. We then have the graded groups $G(A) = \oplus_{n=0}^{\infty} A_n/A_{n+1}, G(B) = \oplus_{n=0}^{\infty} B_n/B_{n+1}$ and the completed groups $\hat{A}, \hat{B}$ defined as above relative to the inverse systems $A/A_n, B/B_n$. Let $\phi : A \to B$ be a homomorphism of filtered groups, so that ϕ restricts (by definition) to a homomorphism from A_n to B_n for all n. Then ϕ induces natural homomorphisms $G(\phi)$ and $\hat{\phi}$ taking $G(A)$ to $G(B)$ and $\hat{A}$ to $\hat{B}$, respectively.

Lemma

With notation as above, if $G(\phi)$ is injective, so is $\hat{\phi}$; likewise, if $G(\phi)$ is surjective, so is $\hat{\phi}$.

Proof.

There are exact sequences $0 \to A_n/A_{n+1} \to A/A_{n+1} \to A/A_n \to 0$ and $0 \to B_n/B_{n+1} \to B/B_{n+1} \to B/B_n \to 0$; they are linked by maps $G_n(\phi) : A_n/A_{n+1} \to B_n/B_{n+1}, \phi_{n+1} : A/A_{n+1} \to B/B_{n+1}, \phi_n : A/A_n \to B/B_n$ in such way that the resulting diagram commutes. The Snake Lemma then yields an exact sequence $0 \to \ker G_n(\phi) \to \ker \phi_{n+1} \to \ker \phi_n \to (B_n/B_{n+1})/\operatorname{im} G_n(\phi) \to (B/B_{n+1})/\operatorname{im} \phi_{n+1} \to \cdots$. It follows by induction on n that all ϕ_n have kernel 0 in the first case, as desired, and all ϕ_n are surjective in the second case; we also have that the map from $\ker \phi_{n+1}$ to $\ker \phi_n$ is surjective in the latter case. Taking inverse limits and using the surjectivity of the inverse systems we get the second assertion. $\square$

The next result is the main step in showing that $\hat{R}$ is Noetherian whenever R is.

Proposition

Let R be a ring, I an ideal of R, M an R-module, and (M_n) an I-filtration of M. Suppose that R is complete in the I-adic topology and that M is Hausdorff in the filtration topology (so that $\cap_n M_n = 0$). Suppose also that the graded module $G(M)$ is finitely generated over $G(R)$. Then M is finitely generated over R.

Proof.

Choose a finite set of generators of $G(M)$ and split them up into their graded components $\alpha_1 \ldots, \alpha_r$, where α_i has degree $n(i)$ and is the image of $x_i \in M_{n(i)}$. Let F^i be the module R with the stable I-filtration given by $F^i_k = R$ if $k \leq n(i)$, $F^i_k = I^{k-n(i)}$ if $k > n(i)$ and put $F = \oplus_{i=1}^r F^i$. Mapping the generator 1 of each F^i to x_i defines a homomorphism $\phi : F \to M$ of filtered groups and $G(\phi) : G(F) \to G(M)$ is a homomorphism of $G(R)$-modules, which by the construction is surjective. By the previous result the inverse limit map $\hat{\phi}$ is surjective. We now have maps $\phi : F \to M, \hat{\phi} : \hat{F} \to \hat{M}$ and maps $\alpha : F \to \hat{F}, \beta : M \to \hat{M}$ making the obvious diagram commute. Since F is free and $R = \hat{R}$ it follows that α is an isomorphism; since M is Hausdorff, β is injective. The surjectivity of $\hat{\phi}$ them implies the surjectivity of ϕ, which says exactly that the x_i generate M over R. $\square$

Corollary

With hypothesies as above, if $G(M)$ is a Noetherian $G(R)$-module, then M is a Noetherian R-module.

I must show that every submodule M' of M is finitely generated. Letting $M'_n = M' \cap M_n$, then (M'_n) is an I-filtration of M' and the embedding $M'_n \to M_n$ gives rise to an injection $M'_n/M'_{n+1} \to M_n/M_{n+1}$ and thus an injection from $G(M')$ into $G(M)$. Then $G(M')$ is finitely generated over $G(R)$ and M' is Hausdorff since M is, whence by the previous result M' is finitely generated.

Finally we get the main result.

Theorem

If R is Noetherian and I is an ideal, then the completion $\hat{R}$ is Noetherian.

Indeed, we have seen that $G_I(R) \cong G_{\hat{I}}(\hat{R})$ is Noetherian. Applying the previous corollary to the complete ring $\hat{R}$, taking $M = \hat{R}$ (filtered by $\hat{I}^n$, so Hausdorff by the construction) we get the desired result. In particular, any power series ring $k[[x_1, \ldots, x_n]]$ over a field k is Noetherian, being the completion of the polynomial ring $k[x_1, \ldots, x_n]$ in the $(x_1, \ldots, x_n)$-adic topology.

Next time I will turn to the proofs of Hensel's Lemma (promised last quarter) and the Cohen Structure Theorem, which shows that completions of coordinate rings in algebraic geometry are closely related to power series rings.