Lecture 5-5: Flatness of completions and Krull's Theorem

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I continue to follow Chapter 10 of Atiyah-Macdonald. Last time I defined the completion $\hat{M}$ of a module M over a ring R with respect to an ideal I; this is a module over the completion $\hat{R}$ of R itself.

I observed last time that rather than using the standard filtration ($M_n = I^n M$) of M (with respect to I) I could use any stable I-filtration (M'_n) (such that $IM'_n \subset M'_{n+1}$ with equality for sufficiently large n) in place of (M_n): any two stable I-filtrations define the same topology on M and $\hat{M}$ is the completion of M with respect to this topology. In particular, $\hat{R}$ could also be defined using any stable I-filtration of R.

Recall now the Artin-Rees lemma from the lecture on April 21: it asserts that if (M_n) is a stable I-filtration of a finitely generated module M over a Noetherian ring R and if M' is a submodule of M, then $(M'n = M' \cap M_n)$ is a stable I-filtration of M'. Combined with the exactness of the inverse limit for surjective inverse systems proved last time, I get

Theorem

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated modules over a Noetherian ring R and if I is an ideal of R, then the sequence $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ is exact.

I now relate completions of modules to tensor products, in particular returning to the notion of flatness, which I treated only sketchily in the fall. Recall that a module M over a commutative ring R is called flat if the tensor product functor $\cdot \otimes_R M$ is exact. If I is an ideal, of R and M is an R-module, then there are natural maps $R \to \hat{R}, M \to \hat{M}$ from R and M to their completions and accordingly a natural map $\hat{R} \otimes_R M \to \hat{R} \otimes_R \hat{M} \to \hat{R} \otimes_{\hat{R}} \hat{M} = \hat{M}$. In general, this last map is neither injective nor surjective, but one has

Theorem

If M is finitely generated then the map $\hat{R} \otimes_R M \to \hat{M}$ is surjective. If moreover R is Noetherian then this map is an isomorphism.

Proof.

It is clear that I-adic completion commutes with finite direct sums. Hence if $F \cong R^n$ then $\hat{R} \otimes_R F \cong \hat{F}$. If M is finitely generated then we have an exact sequence $0 \to N \to F \to M \to 0$ for a suitable F. □

Proof.

This gives rise to an exact sequence $\hat{R} \otimes_R N \to \hat{R} \otimes_R F \to \hat{R} \otimes_R M \to 0$, by the right exactness of $\cdot \otimes_R$, and a sequence $0 \to \hat{N} \to \hat{F} \to \hat{M} \to 0$, together with maps γ, β, α from the first three terms of the first sequence to the second three terms of the second one, making the obvious diagram commute. The map $\delta : \hat{F} \to \hat{M}$ is surjective. A simple diagram chase shows that $\alpha : \hat{R} \otimes_R M \to \hat{M}$ is surjective. If R is Noetherian then the second sequence is exact and it is easy to see that α is an isomorphism. □

Thus the functor sending an R-module M to $\hat{R} \otimes_R M$ is exact on finitely generated R-modules M if R is Noetherian, whence from the definition of the tensor product it is exact on all R-modules. (The functor sending M to $\hat{M}$, by contrast, need not be exact on non-finitely generated modules).

Here are some more properties of the completion $\hat{R}$ of a Noetherian ring R with respect to an ideal I.

Theorem

1. $\hat{I} = \hat{R}I \cong \hat{R} \otimes_R I$;
2. $(I^n)\hat{} = (\hat{I})^n$;
3. $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$;
4. $\hat{I}$ is contained in the Jacobson radical of $\hat{R}$.

Proof.

The first assertion follows from the previous result since I is finitely generated and $\hat{R}I$ is the range of the map from $\hat{R} \otimes_R I$ to $\hat{I}$. The second assertion follows from the first. Then we get $R/I^n \cong \hat{R}/\hat{I}^n$ by the exactness of inverse limits for surjective systems, as above, from which the third assertion follows by taking quotients. I showed in class that the series for $(1 - xy)^{-1} = \sum_i (xy)^i$ converges in $\hat{R}$ for any $x \in I, y \in \hat{R}$ whence $1 - xy$ is invertible in $\hat{R}$ and x lies in the Jacobson radical. $\square$

It follows at once that if R is Noetherian local with maximal ideal I, then the completion $\hat{R}$ is again local with maximal ideal $\hat{I}$. Now I can identify the kernel of the natural map $M \to \hat{M}$, where R is Noetherian, M is a finitely generated R-module, and $\hat{M}$ is its completion with respect to an ideal I of R. The construction shows that this kernel K is the intersection $\cap_n I^n M$ of all the submodules $I^n M$.

Krull's Theorem

With notation as above, K consists of exactly the $m \in M$ with $(1 - i)m = 0$ for some $i \in I$.

Proof.

One direction is obvious: if $(1-i)m = 0, m = im$, then $m = im = i^2 m = \cdots \in K$. Now clearly $IK = K$; since K is finitely generated, say by $k_1, \ldots, k_n$, we can set up an $n \times n$ matrix A whose ith column consists of the coefficients when k_i is expressed as an I-linear combination of k_j. This matrix acts as the identity on the span K of the k_i; by the Cayley-Hamilton Theorem, there is $\alpha \in I$ with $(1-\alpha)K = 0$. The conclusion follows. □

It follows that if S is the multiplicatively closed set $1 + I$, then the kernels of the maps from R to $S^{-1}R$ and from R to $\hat{R}$ coincide, so that one can regard $S^{-1}R$ as a subring of $\hat{R}$. Also if R is a Noetherian domain and I is an ideal then $\cap_n I^n = 0$, since $1 + I$ has no zero divisors. Furthermore, if I is contained in the Jacobson radical of R, then the I-topology on a finitely generated R-module is Hausdorff, since $1 + I$ consists of units in that case. This holds in particular if R is local and I is the unique maximal ideal.