Lecture 5-28: Linear algebraic groups

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I will round out the course with a brief account of linear algebraic groups (I gave a topics course on these in the fall of 2023). Besides providing examples of some of the most important and interesting affine varieties, this will tie together the group theory with which I began the whole sequence last fall and the algebraic geometry with which I am ending this sequence. Throughout I fix an algebraically closed field k.

The basic definition is

Definition

A *linear algebraic group* is an affine variety $G \subset k^n$ which also has a group structure, such that multiplication from $G \times G$ to G and inversion from G to itself are morphisms of varieties.

The most fundamental example is the general linear group $G = GL_n(k)$ of invertible $n \times n$ matrices over k. It takes a bit of work to see that this is an affine variety, since it is defined by an inequality (nonzero determinant) rather than a family of equalities. This is overcome as usual by introducing another variable y, in addition to the n^2 variables x_{ij} for $1 \leq i, j \leq n$ corresponding to the entries of an $n \times n$ matrix M, and then defining G by the equation $(\det M)y = 1$. Then G is also (isomorphic to) an affine open subset of k^{n^2}.

With the example of $GL_n(k)$ in place, one can give many examples of subgroups defined by polynomial equations which are therefore algebraic as well. For instance, we have $SL_n(K)$, the group of $n \times n$ matrices of determinant 1, the group $O_n(k)$ of orthogonal matrices M (such that $MM^t = I$), and the subgroup $SO_n(k)$ of matrices in $O_n(k)$ of determinant 1. Somewhat less familiar, but very important, is the symplectic group $Sp_{2n}(k)$ of all matrices $M \in GL_{2n}(k)$ with $M^t JM = J$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the $2n \times 2n$ matrix having a copy of the $n \times n$ identity matrix I_n in the upper right corner, $-I_n$ in the lower left corner, and all other entries 0.

Alternatively, we can define $O_n(k)$ to consist of all $n \times n$ matrices M with $f(Mv, Mw) = f(v, w)$ for all $v, w \in k^n$, where $f(v, w) = f((v_1, \ldots, v_n), (w_1, \ldots, w_n)) = \sum_i v_i w_i$ is the ordinary dot product on k^n; such an $f : k^n \times k^n \to k$ is called a nondegenerate symmetric bilinear form and arose when I was doing Galois theory last quarter. Similarly, the symplectic group $Sp_{2n}(k)$ consists of all $2n \times 2n$ matrices M with $g(Mv, Mw) = g(v, w)$ for all $v, w \in k^{2n}$, where $g((v_1, \ldots, v_{2n}), (w_1 \ldots, w_{2n})) = \sum_{i=1}^{n} (v_{2i-1} w_{2i} - v_{2i} w_{2i-1})$; such a $g :: k^{2n} \times k^{2n} \to k$ is called a nondegenerate skew bilinear form. Nondegenerate skew bilinear forms exist only in even dimensions and it turns out that any $g \in Sp_{2n}(k)$ automatically has determinant 1. The groups GL_n, SL_n, O_n, SO_n, and Sp_{2n} are called classical.

There are also the more straightforwardly defined subgroups $B_n \subset GL_n(k)$ of upper triangular matrices, U_n of upper triangular matrices with ones on the diagonal, and D_n of diagonal matrices. This last group is called an (n)-torus; by contrast with topology or geometry, if S^1 is the usual circle group in the complex plane, then $(S^1)^n$ is not an algebraic group (it is not defined by polynomial equations) and so would not be called a torus. Note that the subgroup $U_2 \cong k$, regarding k as an *additive* group, since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$. In fact, this group, often denoted G_a, and the 1-torus k^*, often denoted G_m, turn out to be the only connected one-dimensional algebraic groups.

In general, the irreducible components of an algebraic group G turn out to be the cosets of its connected component G_0 containing the identity and G_0 is normal in G, as it would be for any topological group. Thus these components do not overlap (unlike the situation for a general variety). It is therefore customary to speak of a connected algebraic group rather than an irreducible one. Another nice feature distinguishing algebraic groups from other varieties is that they have no singular points: since the differential of the group action takes the tangent space at one point to the tangent space at any other and the group must have smooth points, all points are smooth.

Moreover, given an algebraic group G, suppose that there is a family $\{G_i : i \in I\}$ of closed connected subgroups of G. Then the subgroup H that they generate is closed and connected and in fact there are finitely many indices $i_i, \ldots, i_m \in I$ with $H = G_{i_1} \ldots G_{i_m}$. One proves this by choosing indices $i_1, \ldots, i_m$ such that the subvariety $G_{i_1} \ldots G_{i_m}$ has maximal dimension. One can use this to show that the classical groups $GL_n(k), SL_n(k), SO_n(k)$, and $Sp_{2n}(k)$ are connected, since these groups are well known to be generated by one-dimensional connected subgroups. For example, since the row operations of adding a multiple of one row to another and multiplying a row by a nonzero scalar are well known to reduce any invertible matrix M to the identity matrix, and to be implemented by multiplying M by a matrix lying in a subgroup of $GL_n(k)$ isomorphic to G_1 or G_m, it follows that $GL_n(k)$ is connected. One can argue similarly for $SL_n(k)$, replacing the mutiplication of one row by a nonzero scalar by the multiplication of two successive rows by the scalars a, a^{-1} for some $a \in k^*$.

For the other groups $SO_n(k)$ and $Sp_{2n}(k)$ one must work a little harder. If $G = Sp_{2n}(k)$, then denote the value $g(v, w)$ of the form g at $(v, w) \in (k^{2n})^2$ by (v, w). We then have $(v, v) = 0$ for every nonzero $v \in k^{2n}$. For every $a \in k$ the symplectic transvection $T_{v,a} : k^{2n} \to k^{2n}$ sending $w \in k^{2n}$ to $w + a(w, v)v$ is then easily seen to preserve the form $(\cdot, \cdot)$ and moreover $T_{v,a}T_{v,b} = T_{v,a+b}$. Thus the transvections $T_{v,a}$ for fixed nonzero v form a subgroup of G isomorphic to G_a. These subgroups turn out to generate G, so that G is connected. For $G = SO_n(k)$, one argues slightly differently. Denote the form $f(v, w)$ by (v, w) as in the symplectic case. Then for every nonzero $v \in k^n$ with $(v, v) = 0$ (we call such a v isotropic) there is an isotropic w with $(v, w) = 1$. For $a \in k^*$ there is a unique matrix $S_{v,a} \in G$ sending v to av, w to $a^{-1}w$, and fixing any vector z with $(z, v) = (z, w) = 0$. The matrices $S_{v,a}$ for fixed v form a subgroup of G isomorphic to G_m; the subgroups $\{T_{v,a} : a \in k^*\}$ then generate G.

On the other hand, the group $O_n(k)$ is not connected; as is well known, it has two components, with its identity component being $SO_n(k)$. For $G = GL_n(k)$ or $SL_n(k)$ there is another important subgroup T consisting of the diagonal matrices in G; we have $T = D_n$ if $G = GL_n(k)$ while $T = D_n \cap SL_n(k)$ if $G = SL_n(k)$. In both cases we call T a maximal torus, since it is a torus in G not contained in any other. For the symplectic group $G = Sp_{2n}(k)$ we use a different realization of G, defining it as the set G' of matrices X with $X^t JX = J$, with the matrix J defined as above; then $G' \cong G$. For $G = SO_n(k)$ we let I'_n be the matrix obtained from the $n \times n$ identity matrix I_n by reversing the order of its rows and define G' as the group of all matrices X with $X^t I'_n X = I'_n$; again $G' \cong G$.

Then in both cases (symplectic and orthogonal) we let T be the set of diagonal matrices in G'; this is again a maximal torus. Next time I will indicate how this maximal torus plays a crucial role in understanding the structure not only of a classical group, but also of all reductive groups (of which the classical groups are particular cases).