Lecture 5-23: Blowups

May 23, 2025

Having spent a fair amount of time on flat modules (or families) over a ring, I now give an important example of a non-flat module, having an exceptional fiber over a ring, which can be used to in effect tame the bad (that is, singular) points on certain varieties, systematically replacing them by less bad points on other varieties. I now switch back to Hartshorne, following pp. 28-9 in section 1.4, but also taking a look at Eisenbud's commutative algebraic presentation of the same construction (on pp. 150-1 of Chapter 5).

In its first instantiation, the basic idea of the construction is to replace a single point on a variety by a copy of projective space, which is used to study the structure of the first variety near the point. Start with affine space $\mathbf{A}^n$ and the point $P = (0, \ldots, 0)$ on it. I will blow up $\mathbf{A}^n$ at P, as follows. Consider the product $Y = \mathbf{A}^n \times \mathbf{P}^{n-1}$; this is a variety which is neither affine nor projective, but it is quasi-projective (open in a larger projective variety). Denote the affine coordinates of $\mathbf{A}^n$ by $x_1, \ldots, x_n$ and the homogeneous coordinates of P^{n-1} by $y_1, \ldots, y_n$. Now take the the closed subvariety X of Y defined by the equations $x_i y_j = x_j y_i$ for all i, j; these equations make sense since they are homogeneous in the y_i. We call X the blowup of $\mathbf{A}^n$ at P. We have a natural projection $\phi : X \to \mathbf{A}^n$ given by projection to the first factor. We then have the following properties of X.

1. If $Q \in \mathbf{A}^n$, $Q \neq P$, then the fiber $\phi^{-1}(Q)$ is a single point. Indeed, the equations defining X specify the y_j uniquely as a point in P^{n-1} if the x_i are not all 0.
2. We have $\phi^{-1}(P) \cong \mathbf{P}^{n-1}$. Indeed, there are no conditions on the y_i for a point $(0, \ldots, 0, y_1, \ldots, y_n)$ to belong to X.
3. The points of $\phi^{-1}(P)$ are in bijection to the lines L through P in $\mathbf{A}^n$. Indeed, this was the original definition given of $\mathbf{P}^{n-1}$.
4. X is irreducible. Indeed, X is the union of $X - \phi^{-1}(P)$ and $\phi^{-1}(P)$. The first piece is isomorphic to $\mathbf{A}^n - P$, so is irreducible; since it is dense in X, X is irreducible.

Next I show how to blow up subvarieties of $\mathbf{A}^n$ at points on them.

Definition, p. 29

If V is a closed subvariety of $\mathbf{A}^n$ passing through P, then the *blowup of V at P* is the closure W of $\phi^{-1}(V - P)$ in X. The morphism $W \to V$ obtained by restricting ϕ to W is also denoted ϕ. The blowup of V at any other point Q of it is accomplished by making a linear change of coordinates sending Q to P.

Example

Let V be the plane cubic curve given by the equation $y^2 = x^2(x+1)$. To blow up V at P, we first blow up all of $\mathbf{A}^2$ at P, obtaining the variety X defined by the equation $xu = ty$, where x, y are affine coordinates and t, u homogeneous ones. It looks like $\mathbf{A}^2$ with the point P replaced by a copy of $\mathbf{P}^1$ corresponding to the slopes of lines through P. Call this $\mathbf{P}^1$ the exceptional curve and denote it by E. We then blow up V by combining the equations $y^2 = x^2(x+1)$ and $xu = ty$. Since $\mathbf{P}^1$ is covered by the open sets $t \neq 0$ and $u \neq 0$, we consider these equations separately. If $t \neq 0$, then we can take $t = 1$ and use u as an affine parameter. We then have the equations $y^2 = x^2(x+1), y = xu$, which together yield $x^2(u^2 - x - 1) = 0$. Thus we get two irreducible components, one defined by $x = y = 0, u$ arbitrary, which is E, and the other by $u^2 = x + 1, y = xu$. This last component is the blowup W of V.

Example

Note that W meets E at the points $u = \pm 1$. These points correspond to the two branches of V at P. The inverse image of the x-axis in X consists of E and one other irreducible curve, defined by $u = 0$ and called the strict transform of the x-axis. The strict transform meets E at the point $u = 0$. Similarly, the strict transform of the y-axis meets E at the point $t = 0, u = 1$. The effect of blowing up is thus to separate out branches of curves passing through P according to their tangent lines at this point. If the tangent lines have different slopes, their strict transforms do not meet at X; instead, they meet at E at points corresponding to the different slopes. Note finally that P is a singular point of V but not of W. We say that W resolves the singularity at P.

More generally, let X be an affine subvariety of $\mathbf{A}^n$ and $Y \subset X$ be a closed subvariety. Let $a_1, \ldots, a_r$ be generators of the coordinate ring $R = k[X]$ of X and let $g_0, \ldots, g_s$ be generators of the vanishing ideal I of Y. The blowup of I in R (Eisenbud, p. 150), is the graded ring $B_I R = R \oplus I \oplus I^2 \oplus \cdots \cong R[tI] \subset R[t]$; this is the same as the ring R^* previously introduced in the lecture on April 21 and used to prove the Artin-Rees Lemma. There is a surjection from the polynomial ring $k[x_1, \ldots, x_r, y_0, \ldots, y_s]$ to $B_I R$ sending x_i to a_i and y_i to $g_i t$; its kernel is easily seen to be homogeneous in the y_i. Thus this kernel corresponds to an algebraic subset Z of $\mathbf{A}^r \times \mathbf{P}^s$, which maps onto X via the projection map into $\mathbf{A}^r$. The set Z is the blowup of Y in X. The projection map is an isomorphism away from the preimage of Y; this preimage, called the exceptional set of the blowup, is the projective variety corresponding to the graded ring $G_I R$.

Thus in effect Z is obtained from X by replacing every point of Y with a copy of $\mathbf{P}^s$ for suitable s. It is easy to check that this construction reduces to the previously defined blowup of $\mathbf{A}^n$ at a point if $X = \mathbf{A}^n$, $Y = P$, or more generally to the blowup of a subvariety X of $\mathbf{A}^n$ passing through P at P.

A beautiful geometric consequence of the exceptional set in a blowup corresponding to the associated graded ring relative to the vanishing ideal of the smaller variety is the following. Let $R = k[x_1, \ldots, x_n]/J, I = (x_1, \ldots, x_n)$ with k algebraically closed Let $X = Z(J)$ be the algebraic set corresponding to J and suppose that $J \subset I$, so that the origin P lies in X. The tangent cone of X at P is the cone composed of all lines that are the limits of secant lines joining P to another point Q of X as $Q \to P$. Its ideal turns out to be generated by the initial terms of the polynomials in J, that is, by the homogeneous component of least degree of every polynomial in J.