Lecture 5-21 Flat modules, concluded

May 21, 2025

I will wrap up Chapter 6 of Eisenbud with an important consequence of the Local Criterion for Flatness and a construction of a flat module over $k[t]$ with k a field.

First I need a lemma.

Lemma 6.10, p. 170

Let R be a ring, M an R-module, $x \in R$ a non-zero-divisor which is also a non-zero-divisor on M. For any $R/(x)$-module N we have $\mathrm{Tor}_i^{R/(x)}(N, , M/xM) = \mathrm{Tor}_i^R(N, M)$ for all $i > 0$.

Proof.

Let (F_n) be a free resolution of M as an R-module. I claim that $(R/(x) \otimes F_n)$ is a free $R/(x)$ resolution of $R/(x) \otimes M$. Given this, we may compute $\mathrm{Tor}_i^{R/(x)}((N, M/xM)$ as the homology of N tensored over $R/(x)$ with this resolution, which coincides with N tensored with the resolution of M, and we are done. The homology of $(R/(x) \otimes F_n = F_n/xF_n$ is $\mathrm{Tor}^R((R/(x), M)$; as shown in the lecture on May 16, we have $\mathrm{Tor}_0^R(R/(x), M) = M/xM$,$\mathrm{Tor}_i^R(R/(x), M) = 0$ for $i > 0$, since x is a non-zero-divisor on M. This says that $(R/(x) \otimes F_n)$ is indeed a resolution; it is easily seen to be free over $R/(x)$ as well. □

The corollary relating flat R-modules to flat $R/(x)$ ones is then

Corollary 6.9, p. 170

Let R be a local Noetherian ring with maximal ideal I, S a local Noetherian R-algebra with maximal ideal J such that $IS \subset J$. Let M be a finitely generated S-module and $x \in I$ a non-zero-divisor in R which is also a non-zero-divisor on M. Then M is flat over R if and only if M/xM is flat over $R/(x)$.

Proof.

If M is flat over R, then $M/xM = R/(x) \otimes_R M$ is flat over $R/(x)$; more generally, if $R \to R'$ is a homomorphism of rings and M is flat over R, then $R' \otimes_R M$ is flat over R', since $(R' \otimes_R M') \otimes_{R'} N = M \otimes_R N$ for any R'-module N. So suppose that M/xM is flat over $R/(x)$. Let $k = R/I$ be the residue field. Then $\mathrm{Tor}_1^{R/(x)}(k, M/xM) = 0$, whence by Lemma 6.10, $\mathrm{Tor}_1^R(k, M) = 0$, so M is flat by the local criterion. □

An algebro-geometric situation to which the local criterion is often applied is the following one. Let X, Y be affine varieties over an algebraically closed field k and suppose we have morphisms $\phi : X \to Y, \psi : Y \to \mathbf{A}^1$. We say that X is flat over Y if the homomorphism $k[Y] \to k[X]$ makes $k[X]$ a flat $k[Y]$-module. For every $p \in \mathbf{A}^1$ there is a map of fibers $X_p \to Y_p$, where X_p is the preimage of p under $\psi\phi$ and similarly for Y_p. Let x be a coordinate function on $\mathbf{A}^1$ taking the value 0 at p and let $R = k[Y]_{P'}, M = k[X]_{P''}$, the localizations of $k[Y], k[X]$ at the maximal ideals P', P'' corresponding to p', p''. Assuming (as is usually the case in applications) that X and Y map onto n open subset of $\mathbf{A}^1$, the maps $k[x] \to R, k[x] \to M$ corresponding to ψ and $\psi\phi$, respectively, are injections from $k[x]$ into the domains R, M.

Then x is a non-zero-divisor in both R and M, whence by Corollary 6.3 from the lecture on May 16, X and Y are flat over $\mathbf{A}^1$. In this setting Corollary 6.9 says that if the map $X_p \to Y_p$ is flat in a neighborhood of p'' in X_p, then $X \to Y$ is flat in a neighborhood of p'' in X. Thus if the fibers of $X_p \to Y_p$ vary nicely near p'', then the same is true of all fibers of $X \to Y$ near p''.

Given a ring R and an ideal I of R, one can use I and R to produce an R-algebra $\mathcal{R}(R, I)$ called the Rees algebra, similar in spirit to the graded ring $G_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$ defined in the lecture on April 18. Set $\mathcal{R}(R, I) = \sum_{n=-\infty}^{\infty} I^n t^{-n} = R[t, t^{-1}I] \subset R[t, t^{-1}]$ (see p. 171), where $R[t, t^{-1}]$ is the ring of Laurent polynomials in one variable over R; here we take $I^n = R$ if $n \leq 0$. If R is a k-algebra (with k a field) then we regard $\mathcal{R}(R, I)$ as a $k[t]$-algebra. Then $\mathcal{R}(R, I)/t\mathcal{R}(R, I) = G(I), \mathcal{R}(R, I)/(t - a)\mathcal{R}(R, I) = R$ for any $a \in k^*$, so that $\mathcal{R}(R, I)$ has fiber $G_I(R)$ at $t = 0$ and R at $t = a$ for $a \neq 0$. This behavior of the fibers is uniform enough to yield

Corollary 6.11, p. 172

The Rees algebra $S = \mathcal{R}(R, I)$ is flat over $k[t]$. If $\cap_{i=1}^{\infty} I^i = 0$, then every element of the form $1 - ts$ with $s \in S$ is a non-zero-divisor in S.

Proof.

For the first statement it is enough by Corollary 6.3 cited above to show that S is torsion-free as a $k[t]$-module, and this is immediate since $S \subset R[t, t^{-1}]$. For the second statement, note that if $p(1 - ts) = 0$ for some $p \in S$, then looking at the equation modulo t we get $p = qt$ for some $q \in S$, whence $q(1 - st) = 0$ since t is not a zero divisor. Repeating this argument, we see that $p \in t^n S$ for all n. Writing $p = \sum_{i=-j}^{j} p_i t^i$ with $p_i \in R$ we get $p_i \in I^n$ for all n, whence $p = 0$ as required. □

I conclude with two examples, one of them historically the first flat family to be considered. Fix a degree d. For each ordered triple $\alpha = (a_0, a_1, a_2)$ of nonnegative integers a_i summing to d let x_α be an indeterminate. Let $R = k[\{x_\alpha\}]$ be the polynomial ring in the x_α over k, an algebraically closed field, and set $S = R[y_0, y_1, y_2]/I$, where I is the ideal generated by all $\sum_\alpha x_\alpha y^\alpha$, where y^α denotes $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2}$. Geometrically, this corresponds to the family of all projective plane curves of degree d; one could replace 3 by any number $r+1$ and get the family of hypersurfaces of degree d in $\mathbf{P}^r$. Except over the point where all x_α are 0, this family is "good"; algebraically the fiber at a prime ideal P is a polynomial over the quotient field of R/P modulo an equation of degree d and fibers over different points look much the same. Here it turns out that S is not flat as an R-module, but if we invert any x_α, then $S[x_\alpha^{-1}]$ is flat over $R[x_\alpha^{-1}]$.

On the other hand, the blowup of the plane is not flat over the plane. This means that if $R = k[x, y]$ and S is the subring of the quotient field of R generated by x/y and y, then S is not flat over R. Here, flatness fails because the fiber over the origin is a curve, whereas nearby fibers are only points. I will say more about blowups next time.