Lecture 5-2: Completions of rings

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I now return to completions of rings, first discussed in the lecture last quarter on March 7. I start with a very general construction, valid for any abelian topological group, and then specialize to e commutative ring setting. I am following Chapter 10 in Aityah-Macdonald's book Introduction to Commutative Algebra.

To begin with, then, let A be a topological additive abelian group, so that A has a topology making the group operations continuous. In this setting one can define the notion of Cauchy sequence: a sequence (a_i) of elements of A is Cauchy if for every neighborhood U of 0 in A there is in index N such that whenever $i, j \geq N$ we have $a_i - a_j \in U$. Such a sequence converges to $a \in A$ if for every such neighborhood U there is an index M such that whenever $i, \geq M$ we have $a_i - a \in U$. If the topology on A is Hausdorff, then the limit of any convergent sequence is unique.

One can now construct the completion $\hat{A}$ of A; roughly speaking this is the smallest topological group containing A in which all Cauchy sequences converge. The elements of $\hat{A}$ are equivalence classes of Cauchy sequences (a_i) in A, where we decree that $(a_1, a_2, \ldots) \sim (b_1, b_2, \ldots)$ if and only if the interleaved sequence $(a_1, b_1, a_2, b_2, \ldots)$ is Cauchy. It is easy to check that this is indeed an equivalence relation. There is an obvious additive structure on $\hat{A}$, obtained by adding sequences term by term; one can verify that this operation respects the equivalence relation. to make $\hat{A}$ a topological group, we decree that an open neighborhood U' of any equivalence class $[a] = [(a_1, a_2, \ldots)$ of Cauchy sequences is one consisting of all sequences $(b_1, b_2, \ldots)$ such that there is an index N such that $a_i - b_i \in U$ for all $i \geq N$, where U is a fixed neighborhood of 0 in A

Untangling the definition, one finds that a Cauchy sequence $(c_1, c_2, \ldots)$ of Cauchy sequences $c_i = (c_{i,1}, c_{i,2}, \ldots)$ in A is one for which given any neighborhood U of 0 in A there is an index N, such that whenever $i, j, k, \ell \geq N$ we have $c_{i,j} - c_{k,\ell} \in U$. In order to be sure that such a sequence converges, we need to make an additional assumption about A, namely that it is first countable. This means that there is a countable sequence $U_1, U_2, \ldots$ of neighborhoods of 0 in A such that any neighborhood of 0 contains U_i for some i. For every $a \in A$, if we set $U_{a,i} = a + U_i$ then we get a countable sequence of neighborhoods of a such that any neighborhood of a contains $U_{a,i}$ for some i.

Under this assumption, given a Cauchy sequence $(c_1, c_2, \ldots)$ as above, it converges to $(d_1, d_2, \ldots)$, where $d_i = c_{N_i,N_i}$ and N_i is chosen so that $c_{j,k} - c_{j,\ell} \in U_i$ for all $j, k, \ell \geq N_i$. In this way we construct complete topological spaces without using a metric. For example, the group $\mathbb{R}$ under addition can be constructed in this way from $\mathbb{Q}$, taking U_i to be the open interval $(-1/i, 1/i) \cap \mathbb{Q}$.

Now however I want to consider completions of a very different flavor than $\mathbb{R}$. I do this by specializing to the case where the neighborhoods U_i are *subgroups* of A. Note that if such U_i exist then A is highly disconnected, since the subgroups U_i are then simultaneously open (by definition) and closed (since their complements are unions of cosets and thus also open). The topology is Hausdorff if and only if the intersection of all the U_i is the single point 0.

Finally I can return to the setting of the lecture of March 7: given a commutative ring R and an ideal I of R, define a topology on R by decreeing that a subset U is open if and only if it contains $x + I^n$ for some n whenever it contains $x \in R$. The ring operations are then continuous on R. Observe that a sequence (r_i) of elements of R is Cauchy if and only if the image $\bar{r}_i$ of r_i in the quotient R/I^n is eventually constant in i, for all fixed n. Replacing r_i by its image r'_i in R/I^i, one obtains a a sequence $(r'_0, r'_1, \ldots)$ as in the earlier lecture; that is, an element of the inverse limit attached to the inverse system $(R_n = R/I^n)$, reviewed below. The above construction shows that every Cauchy sequence of such sequences converges.

I now head toward the proof that if one completes the same ring R twice with respect to the I-adic topology, the result is the same as completing it once. To do this, I study the exactness properties of inverse limits more generally. Let $\ldots \to A_2 \to A_1 \to A_0$ be a sequence of abelian groups equipped with homomorphisms $f_n : A_n \to A_{n-1}$. This is called an inverse system. One then has the inverse limit $\overleftarrow{A}$ of this system; it consists of all tuples $(a_1, a_2, \ldots)$ such that $a_i \in A_i$ for all i and $f_i(a_i) = a_{i-1}$ for $i \geq 1$. There is an obvious structure of abelian group on $\overleftarrow{A}$. Now suppose that $(B_n), (C_n)$ are two other inverse systems, with maps g_n, h_n, respectively, and let $\overleftarrow{B}, \overleftarrow{C}$ be their inverse limits. Assume that for each n we have a short exact sequence S_n, namely $0 \to A_n \to B_n \to C_n \to 0$ such that the obvious diagram with S_n as its top row and S_{n-1} as its bottom row commutes.

Set $A = \prod_{i=0}^{\infty} A_i, B = \prod_{i=0}^{\infty} B_i, C = \prod_{i=0}^{\infty} C_i$. Define a map $d^A : A \to A$ via $d^A(a_n) = a_n - f_{n+1}(a_{n+1})$ for $a_n \in A_n, a_{n+1} \in A_{n+1}$ and define maps d^B, d^C similarly. Then $\overleftarrow{A}$ is just the kernel of d^A, and similarly for $\overleftarrow{B}, \overleftarrow{C}$. Now we have exact sequences $A \to B \to C \to 0$ and $0 \to A \to B \to C$, together with maps d^A (resp. d^B, d^C) from A to itself (resp. from B, C to themselves) such that an obvious diagram commutes. Then something called the Snake Lemma (see Exercise 17.1.3, p. 792) kicks in and asserts that there is an exact sequence $0 \to \ker d^A \to \ker d^B \to \ker d^C \to A/\text{im } d^A \to \cdots$.

Now the inverse systems $R_n = R/I^n$ that I have in mind have the property that the maps f_n are always onto; such systems are called surjective. Whenever this property holds for the inverse system (A_n) and one is given $(a_1, a_2, \ldots) \in A$, one can choose $x_0 \in A_0$ arbitrarily and solve the equations $x_n - f_{n+1}(x_{n+1}) = a_n$ for $x_1, x_2, \ldots$ inductively. The upshot is that taking the inverse limit of an inverse system is an exact functor when restricted to surjective systems. In particular, given R and the ideal I, the exact sequence $0 \to I^n \to R \to R/I^n \to 0$, coupled with the chains of ideals $I^n \supset I^{n+1} \supset \cdots$ in I^n and $J^0 \supset J^1 \supset \cdots \supset J^n = 0$ in R/I^n, where J is the image of I in R/I^n, show that $R/I^n \cong \hat{R}/\hat{I}^n$ for all n, where $\hat{I}$ is the canonical image of I in $\hat{R}$; here the completion of R/I^n is 0 since every Cauchy sequence is identified with 0. Thus $\hat{R}$ is I-adically complete: its I-adic completion is isomorphic to $\hat{R}$ itself.

We can carry out the same construction with an R-module M in place of R (given R and the ideal I). Thus we can set $M_n = I^n M$ and take the inverse limit $\hat{M}$ of the inverse system (M/M_n). The submodules M_n of M are a countable base at 0 for a topology on M making it a topological R-module (where R has the I-adic topology). Then $\hat{M}$ is the completion of M in this topology, also called the I-adic topology, and it is a $\hat{R}$-module. In fact one has considerable freedom of choice in setting it up. We could replace (M_n) by any stable I-filtration of M in the sense of the lecture on April 21, that is, by a chain of submodules $M'_0 \supset M'_1 \supset \cdots$ such that $IM'_m \subseteq M'_{m+1}$ for all m with equality for sufficiently large m. Any two such filtrations define the same topology on M and accordingly give rise to isomorphic completions of M. The completion $\hat{M}$ of M is itself complete with respect to the I-adic topology.