Lecture 5-19: Flat modules, continued

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Continuing with Chapter 6 of Eisenbud, I give an equational criterion for flatness and use it show that under mild conditions, given a module over a local ring, it suffices to check the ideal criterion for flatness given last time on just the maximal ideal.

I begin by translating the criterion given last time for a sum of tensors to be 0 in a tensor product to a criterion for flatness.

Equational Criterion for Flatness, p. 165

An R-module M is flat if and only if the following condition is satisfied: for every relation $\sum_i n_i m_i = 0$ with $m_i \in M, n_i \in R$ there are elements $m'_j \in M, a_{ij} \in R$ such that $\sum_j a_{ij} m'_j = m_i$ for all i and $\sum_i a_{ij} n_i = 0$ for all j.

Proposition 6.1 last time says that M is flat if and only if for every ideal I, an element $x = \sum_i n_i \otimes m_i \in I \otimes M$ goes to 0 in $R \otimes M$ if and only if $\sum_i n_i \otimes m_i$ satisfies the criterion given last time to be 0. Since the image of x in $R \otimes M = M$ is $\sum_i n_i m_i$, the result follows.

This last criterion can be formulated in terms of a commutative diagram, as follows.

Corollary 6.6, p. 166

A module M over a ring R is flat if and only if for every map $\beta : F \to M$ from a free module F of finite rank and for every submodule K of $\ker \beta$ generated by one element there are maps $\gamma : F \to G, \pi : G \to M$ with $\beta = \pi\gamma$, G free, and $K \subset \ker \gamma$. Equivalently, M is flat if and only if this condition holds for every finitely generated submodule of $\ker \beta$. If M is finitely presented, then M is flat if and only if M is projective.

Proof.

This is just a translation of the Equational Criterion. An element f in the kernel of a map from a free module F to M is a relation on the images $m_i \in M$ of basis elements of F. The elements m'_j of the preceding result correspond to a map from another free module G taking the generators of G to the m'_j. A matrix with entries a_{ij} such that $\sum_j a_{ij} m'_j = m_i$ corresponds to a map γ as in the statement; the condition that $\sum_i a_{ij} n_i = 0$ for all j says that $\gamma(f) = 0$. If the condition holds for submodules K generated by single elements, then by composing finitely many maps γ killing particular elements of $\ker \beta$, we get such a γ killing any finite subset of $\ker \beta$. □

Proof.

Finally, I proved already in the fall that projective modules are flat. If M is finitely presented, than (by definition) there is a surjection from a finitely generated free module F onto M whose kernel K is also finitely generated; choosing γ as in the second assertion, its image is carried isomorphically onto M by the map from G, so that the map $G \to M$ splits and M is a direct summand of a free module, hence projective. □

I now prove a result asserting that a nice property of one fiber of a family implies a nice property of the family itself.

Corollary 6.7, p. 167

Let k be a field and $R = k[t]$; let S be a Noetherian ring flat over R. If the fiber S/tS over the prime (t) is a domain and U is the set of elements of the $1 - ts$ for $s \in S$, then the localization $S[U^{-1}]$ of S at U is a domain.

Proof.

Since localization preserves flatness, we can replace S by $S[U^{-1}]$ at the beginning and assume that all elements of the form $1 - ts$ are already units of S. Suppose that $I, J \subset S$ are ideals with $IJ = 0$; we must show that $I = 0$ or $J = 0$. Enlarging I and J if necessary, we may assume that each is the annihilator of the other. Since $IJ \equiv 0 \mod (t)$ and $S/(t)$ is a domain, we may assume that $J \subset (t)$. Then $J = (J : t)t$, where $(J : t)$ denotes $\{x \in S : xt \subset J\}$. Since t is a non-zero-divisor of S (by flatness of S) and $I(J : t)t = 0$, we get that $(J : t)$ annihilates I, forcing $(J : t) \subset J, J = Jt$. Since J is finitely generated an application of the Cayley-Hamilton Theorem shows that $(1 - ts)J = 0$ for some $s \in S$, forcing $J = 0$, as desired. $\square$

Note that the localization is essential in the last result; it is not always true that the ring S itself is a domain. For example, if $R = k[t], S = k[x,t] \times k[t,t^{-1}]$ with k a field, then the fiber over $(t-a)$ is $S/(t-a)$ for $a \in k$; when $a = 0$, this is a domain, since $tk[t,t^{-1}] = k[t,t^{-1}]$, but for $a \neq 0$ the fiber is not a domain (and neither is S itself). Such pathologies can be avoided by working with graded rings (or geometrically with projective maps).

Now I can prove the main result, giving a criterion for flatness using just the maximal ideal of a local ring rather than all ideals.

Local Criterion for Flatness, p. 168

Let R be a local Noetherian ring with maximal ideal I and S a local Noetherian R-algebra with maximal ideal J. Assume that $IS \subset J$. If M is a finitely generated S-module, then M is flat over R if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$.

Proof.

We know that if M is flat, then $\mathrm{Tor}_1^R(R/I, M) = 0$. Now suppose that S and M are as in the theorem and that the hypothesis on Tor_1^R is satisfied. I first show that $\mathrm{Tor}_1^R(N, M) = 0$ if N has finite length over R, by induction on the length. The case of length 1 follows from the hypothesis, since R/I is the only module of length 1 over R. The long exact sequence for Tor and the inductive hypothesis then proves the vanishing for any N of finite length. Now let K be an arbitrary ideal and suppose that $u \in K \otimes M$ is in the kernel of the multiplication map from $K \otimes M$ to M. I will show that $u = 0$. The S-module structure on M gives $K \otimes M$ an S-module structure and we have $I^n(K \otimes M) \subset J^n(K \otimes M)$. Since $K \otimes M$ is finitely generated over S, Krull's Theorem on the kernel of the map from a Noetherian ring to its completion implies that $\cap_n J^n(K \otimes M) = 0$, whence $\cap_n I^n(K \otimes M) = 0$. Hence it suffices to show that $u \in I^n(K \otimes M)$ for every n. □

Proof.

The module $I^n(K \otimes M)$ is the image in $K \otimes M$ of $(I^n K) \otimes M$. By the Artin-Rees lemma, $I^t \cap K \subset I^n$ for sufficiently large t, so it suffices to show that u lies in the image of $(I^t \cap K) \otimes M$ for all t. Tensoring the short exact sequence $0 \to I^t \cap K \to K \to K/(I^t \cap K) \to 0$ with M produces the exact sequence $(I^t \cap K) \otimes M \to K \otimes M \to K/(I^t \cap K) \otimes M \to 0$, so it suffices to show that u goes to 0 in $K/(I^t \cap K) \otimes M$. The map $K \otimes M \to K/(I^t \cap K) \otimes M$ is obtained by tensoring the map $K \to K/(I^t \cap K)$ with M; here K embeds in R while $K/(I^t \cap K) \otimes M$ maps to $R/I^t \otimes M$, say by the map $\phi \otimes 1$. Now it suffices to show that the kernel of $\phi \otimes 1$ is 0. □

Proof.

Identifying $K/(I^t \cap K)$ with $(K + I^t)/I^t$, we see that ϕ is the leftmost map in the exact sequence
$0 \to (K + I^t)/I^t \to R/I^t \to R/(K + I^t) \to 0$. Applying Tor, we get a long exact sequence of which a part is
$\mathrm{Tor}_1^R(R/(K + I^t), M) \to (K + I^t)/I^t \otimes M \to R/I^t \otimes M$, where the rightmost map is $\phi \otimes 1$, so it is enough to show that
$\mathrm{Tor}_1^R((R/K + I^t), M) = 0$. Since $R/(K + I^t)$ is annihilated by I^t, it has finite length as an R-module, so we are done by the first part of the proof. □