Lecture 5-16: Flat modules

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I return now to Eisenbud, this time starting in Chapter 6. Starting with a pair of rings R, S with $R \subset S$, so that S is an R-algebra, the idea is to see how the quotient S/PS behaves as P varies over prime ideals in R. The collection of quotients S/PS, or just the single ring S, is called a family; a particular quotient S/PS is called the fiber over P (viewing P as a point in the Zariski topology of R). The nicest behavior will occur when S is flat over R; recall that this means that the functor $\cdot \otimes_R S$ is exact, or equivalently just left exact. If P is a prime ideal of R, denote by K_P the quotient field of the integral domain R/P.

I begin with some examples. Throughout let k be an algebraically closed field of characteristic 0 and set $R = k[t]$. First let $S = R[x]/(x - t)$ (Example 1, p. 159). Here $S \cong R$, so clearly $S/MS \cong R/M$ for any M, so the fibers are as uniform as one could ask. Notice that S is flat as an R-module, since $S \otimes_R N = N$ for any R-module N. Next let $S = R[x]/(x^2 - t)$ (Example 2). Here the fiber over a point $P = (t - a)$ with $a \in k^*$ is $k[x]/(x^2 - a) \cong k \oplus k$, while the fiber over (t) is $k[x]/(x^2)$. Finally, the fiber over 0 is $k(t)[x]/(x^2 - t)$, a field of degree 2 over the residue field $K_0 = k(t)$. Thus for each prime P the fiber over P is a vector space of dimension 2 over the residue field K_P. Here S is a free R-module with basis $(1, x)$, so again S is flat.

Next set $S = R[x]/(tx - 1) \cong R[t^{-1}]$ (Example 3, p. 160); clearly we are localizing the algebra S of Example 1. It is not difficult to check that any localization of a ring is flat as a module over the ring, so S is again flat. The fiber over a prime P is K_P, corresponding to one point, except when $P = (t)$, in which case this fiber is the 0 ring, corresponding to the empty variety. In this case S is not a free R-module; accordingly there is some variation in the fibers, but this variation is not too drastic. Finally, let $S = R[x]/(tx - t)$. Here S is *not* flat over R since it has t-torsion; I will rule out this behavior for flat modules shortly. If the prime ideal P does not contain t, then t is a unit in K_P, whence $K_P \otimes_R S \cong K_P$, corresponding to a point; but if $P = (t)$, then $tx - t = 0$ in $K_P \otimes_R R[x]$ and $K_P \otimes_R S = k[x]$, corresponding to a line. In general, it turns out that S is never flat over R if the dimension of one fiber is greater than that of nearby fibers.

I mentioned the Tor functor briefly in the fall; let me now give a slightly more systematic account, emphasizing the parallels between this functor and the Ext functor. To begin with, for any R-modules M, N the R-module $\mathrm{Tor}_n^R(M, N)$ is the nth higher derived functor of the functor $\cdot \otimes_R N$, so that $\mathrm{Tor}_0^R(M, N) = M \otimes_r N$, while in general $\mathrm{Tor}_n^R(M, N)$ is computed by letting $\ldots \to F_1 \to F_0 \to M$ be a free resolution of M and taking the homology of the chain complex $\ldots F_{i+1} \otimes N \to F_i \otimes N \to \ldots$, so that $\mathrm{Tor}_n^R(M, N)$ is the kernel of the map $F_i \otimes_R N \to F_{i-1} \otimes_R N$ modulo the image of the map $F_{i+1} \otimes_R N \to F_i \otimes_R N$. As with Ext groups, this homology is independent of the choice of the resolution (F_i); it could also be computed by resolving N rather than M. The functor $\mathrm{Tor}_n^R(\cdot, N) =$$\mathrm{Tor}_n^R(N, \cdot)$ is covariant.

If M or N is free over R, then $\mathrm{Tor}_n^R(M,N)=0$ for $n>0$; if M and N are finitely generated and R is Noetherian, then $\mathrm{Tor}_n^R(M,N)$ is finitely generated. Given a short exact sequence $0\to M'\to M\to M''\to 0$ of R-modules and another R-module N, one gets a long exact sequence $\cdots$ $\mathrm{Tor}_i^R(M',N)\to\mathrm{Tor}_i^R(M,N)\to\mathrm{Tor}_i^R(M'',N)\to\cdots\to M'\otimes_R N\to M\otimes_R N\to M''\otimes_R N\to 0$; in particular, M is flat if and only if $\mathrm{Tor}_n^R(M,N)=0$ for $n>0$. As a simple example of an explicit computation of Tor, let $x\in R$ be a non-zero-divisor and let M be an R-module. I will compute $\mathrm{Tor}_i^R(R/(x),M)$. The short exact sequence $0\to R\to R\to R/(x)\to 0$, in which the second map is multiplication by x, is a free resolution of $R/(x)$; tensoring with M we get $\mathrm{Tor}_0^R(R/(x),M)=M/xM$, $\mathrm{Tor}_1^R(R/(x),M)={}_xM=\{m\in M: xm=0\}$, $\mathrm{Tor}_i^R(R/(x),M)=0$ for $i>1$. In what follows I will often omit the subscript R on tensor product of R-modules.

A simple application of the long exact sequence for Tor is the following. It is the analogue for Tor of the Baer Criterion seen in the fall for injectivity.

Proposition 6.1, p. 162

Let R be a ring, M an R-module, and I an ideal of R. The multiplication map $I \otimes_R M \to M$ is an injection if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$. The module M is flat if and only if this condition is satisfied for every finitely generated ideal I.

Proof.

The short exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to a long exact sequence including $\mathrm{Tor}_1^R(R, M) \to \mathrm{Tor}_1^R(R/I, M) \to I \otimes M \to R \otimes M$. The leftmost term is 0, by above results, the rightmost term is M,and the rightmost map is multiplication, so the first assertion follows. If the hypothesis in the second assertion holds, then the map $I' \otimes M \to M$ is an injection for any ideal I', since the definition of the tensor product guarantees that any element of $I' \otimes M$ is a finite sum of tensors involving only finitely many elements of I'. Similarly, the statement that x$\in$ $N' \otimes M$ lies n the kernel of the map from this module to $N \otimes M$ involves only finitely many elements of N, so we are reduced to the case where N is finitely generated. □

Proof.

In this case we have a chain of submodules $N' = N_0 \subset N_1 \subset \cdots \subset N_p = N$, where each quotient N_i/N_{i-1} is singly generated and so isomorphic to a quotient R/I. Repeated application of the hypothesis and the first assertion shows that the map $N_i \otimes M \to N_{i+1} \otimes M$ is an injection for all i, and flatness of M follows. □

Next I give two simple consequences, again analogous to corresponding results for injective modules.

Corollary 6.2, p. 163

Let k be a field and R the ring $k[t]/(t^2)$. If M is an R-module then M is flat if and only if multiplication by t induces an isomorphism from M/tM to tM.

This follows at once from the last two results, since (t) is the only nonzero ideal of R.

Corollary 6.3, p. 164

If $a \in R$ is a non-zero-divisor and M is flat over R, then a is a non-zero-divisor on M. If R is a PID, then M is flat over R if and only if it is torsion-free.

The first assertion follows at once from Proposition 6.1 applied to the principal ideal generated by a. The second one follows from the same proposition and the above computation of $\mathrm{Tor}_1^R(R/(a), M)$.

The last result can be improved to a general criterion for flatness. To do this I need a criterion for a combination of tensors to equal 0 in a tensor product.

Lemma 6.4, p. 164

Let M, N be R-modules and suppose that N is generated by a set $\{n_i\}$ of elements. Every element of $M \otimes N$ may be written as a finite sum $\sum_i m_i \otimes n_i$ in $M \otimes N$. Such an expression is 0 if and only if there are elements $m'_j \in M$ and $a_{ij} \in R$ such that $\sum_j a_{ij} m'_j = m_i \in M$ for all i while $\sum_i a_{ij} n_i = 0 \in N$ for all j.

Proof.

If elements m'_j and a_{ij} exist with this property, then $\sum_i m_i \otimes n_i = 0$, as one sees by moving the a_{ij} past the tensor product. To prove the converse, suppose first that the n_i are a free basis of N. You saw in the fall that $\sum_i m_i \otimes n_i = 0$ if and only if all $m_i = 0$, so that we can take $a_{ij} = 0$ for all i, j. In general, let G be a free R-module on a set $\{g_i\}$ of generators in bijection to $\{n_i\}$ and let F be a free module surjecting onto the kernel of the surjection from G to N. Right exactness of the tensor product shows that the sequence $M \otimes F \to M \otimes G \to M \otimes N \to 0$ is exact; it sends $\sum m_i \otimes g_i$ to 0. Then $\sum_i m_i \otimes g_i = \sum_j m'_j \otimes y_j$ for some $m'_j \in M$ with the y_j lying in the image of F, so that $y_j \to 0$ in N. Writing each y_j as a combination $\sum_i a_{ij} g_i$ of basis elements and using the special case above on the difference $0 = \sum_i m_i \otimes g_i - \sum_j m'_j \otimes \sum_i a_{ij} g_i$, we get that $m_i = \sum_j a_{ij} m'_j$ and $y_j = \sum_i a_{ij} g_i$ goes to $\sum_i a_{ij} n_i = 0$, as required. □