Lecture 5-14: Intersections in projective space

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Following section 1.7 of Hartshorne, I will study how a projective variety intersects a hypersurface in $\mathbf{P}^n$ not containing it, attaching a suitable degree to the variety and multiplicities to the irreducible components of the intersection.

I begin by studying the dimension of intersections of subvarieties of affine space. Before I do this I need a simple construction to produce new varieties from old ones. Given two affine algebraic sets $X \subset \mathbf{A}^n$, $Y \subset \mathbf{A}^m$, their Cartesian product $X \times Y \subset \mathbf{A}^{n+m}$ is an affine algebraic set in an obvious way. Its topology is the one induced from $\mathbf{A}^{n+m}$ and is *not* generally the same as the product topology. It is an easy exercise to show that if X and Y are irreducible, then so is $V = X \times Y$; by looking at chains of subvarieties of V, one checks that $\dim X \times Y = \dim X + \dim Y$. More generally, the product of two schemes has a scheme structure. If $X \subset \mathbf{P}^n$, $Y \subset \mathbf{P}^m$, then one *cannot* however embed $X \times Y$ in $\mathbf{P}^{n+m}$, due to the vagaries of homogeneous coordinates. Instead, if $[x_0, \ldots, x_n] \in \mathbf{P}^n$, $[y_0 \ldots, y_m] \in \mathbf{P}^m$, then $[x_0y_0, \ldots, x_ny_0, x_0y_1, \ldots, x_ny_1, \ldots, x_ny_m]$ is well defined as a point in $\mathbf{P}^{n+m+nm}$. In this way we realize $X \times Y$ as a subvariety of $\mathbf{P}^{n+m+nm}$. This is called the Segre embedding. More generally, the product of two projective varieties is again a projective variety.

Then we have

Affine Dimension Theorem. p. 48, Hartshorne

Let Y, Z be varieties of dimensions r, s in $\mathbf{A}^n$. Then every irreducible component W of $Y \cap Z$ has dimension at least $r + s - n$.

Proof.

First suppose Z is a hypersurface, defined by an equation $f = 0$. If $Y \subseteq Z$, there is nothing to prove; otherwise the result follows from the corollary to Krull's theorem observed in the lecture on April 23. In general, the product $Y \times Z \subset \mathbf{A}^{2n}$ has dimension $r + s$. The diagonal $\Delta = \{(P, P) : P \in \mathbf{A}^n\}$ is also a subvariety of $\mathbf{A}^{2n}$ isomorphic to $\mathbf{A}^n$. In the isomorphism $Y \cap Z$ corresponds to $Y \times X \cap \Delta$. In this way we reduce the result to the varieties $Y \times Z$ and Δ. But Δ is an intersection of n hyperplanes, each defined by equating two coordinates. Applying Krull's Theorem n times, we get the result. $\square$

Given a fixed component of $Y \cap Z$, some of the hyperplanes in the preceding proof will contain it and others will not; this is why (in contrast to Krull's Theorem) not all components need have the same dimension.

For projective varieties we get more uniform behavior. This is

Projective Dimension Theorem, p. 48

If Y, Z are subvarieties of $\mathbf{P}^n$ of dimensions r and s, then all irreducible components of $Y \cap Z$ have dimension at least $r + s - n$. Moreover, if $r + s - n \geq 0$, then $Y \cap Z$ is nonempty.

The first statement follows from the previous theorem, since $\mathbf{P}^n$ is covered by affine open sets. To prove the second statement, replace Y, Z by their affine cones $C(Y), C(Z)$ (see the lecture on April 16), which have the respective dimensions $r + 1$ and $s + 1$. If $r + s \geq n$, then $C(Y)$ and $C(Z)$ must intersect in at least a line, whence $Y \cap Z \neq \emptyset$.

Now I need a small refinement of earlier results on associated prime ideals of finitely generated modules over a Noetherian ring; see the lecture on April 11. I showed then that every finitely generated module M over a Noetherian ring R admits a finite chain of submodules $M_0 = 0 \subset M_1 \subset \cdots \subset M_n = M$ such that each quotient $M_i/M_{i_1} \cong R/P_i$ for some prime ideal P_i. The refinement I need now is that in any two such chains, the number n_P of quotients isomorphic to R/P for a fixed prime ideal P minimal among the P_i is the same; in fact it equals the length of the localization M_P of M at P as a module over R_P. This number n_P is called the multiplicity of M at P and is denoted $\mu_P(M)$. This is easily checked.

A further refinement needed in the projective setting is

Lemma, p. 50

With notation as above, if both R and M are graded, then we can choose the submodules M_i in the chain to be graded and each quotient M_i/M_{i-1} takes the form $(R/P_i)(\ell_i)$ for some homogeneous prime ideal P_i, where $(R/P_i)(\ell_i)$ denotes the graded ring R/P_i with the degrees of all graded components shifted down by $\ell_i \in \mathbb{Z}$.

This is proved as for the previous result, taking account of the degrees of the elements m_i whose annihilators are the prime ideals P_i arising in the statement.

Recall also from the lecture on April 18 the Hilbert polynomial $P_M = \ell(M_n)$, the length of M_n as a module over R_0 (which is a polynomial in n for sufficiently large n), where $M = \oplus_{n=0}^{\infty} M_n$ is a finitely generated graded module over a graded Noetherian ring $R = \oplus_{n=0}^{\infty}$ and the 0-graded piece R_0 is Artinian. If this polynomial has degree r, then its coefficients are integers divided by r! (Note that the degree of this polynomial is one less than the integer $d(M)$ attached to M at the beginning of the lecture on April 18. This shift works well in the projective setting, where the dimension of a projective variety V with corresponding ideal $I \subset k[x_0, \ldots, x_n]$ is one less than the dimension of $k[x_0, \ldots, x_n]/I$.) We define the degree of M to be r! times its leading coefficient (p. 52). If $Y \subset \mathbf{P}^n$ is a projective variety, then we define its degree $\deg Y$ to be the degree of the homogeneous coordinate ring $k[Y]$, regarding the latter as a graded module over the polynomial ring $k[x_0, \ldots, x_n]$. The integer r arising in the definition of degree is of course just the dimension of Y as a projective variety.

Proposition, p. 52

1. If Y is nonempty, then its degree is a positive integer.
2. If $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 has the same dimension r and $Y_1 \cap Y_2$ has lesser dimension, then $\deg Y = \deg Y_1 + \deg Y_2$.
3. $\deg \mathbf{P}^n = 1$.
4. If H is a hypersurface defined by a homogeneous polynomial of degree d, then $\deg H = d$.

Proof.

If Y is nonempty, then the Hilbert polynomial of its coordinate ring is a nonzero polynomial of degree $r = \dim Y$. The degree of $k[Y]$ is then an integer, by above remarks, and is positive since it equals the dimension of a graded component up to a scalar. For the second assertion, we have that I, the ideal of Y, is the intersection $I_1 \cap I_2$ of the ideals I_1, I_2 of Y_1, Y_2. We have an exact sequence $0 \to S/I \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0$ for a suitable polynomial ring S and the zero set of $I_1 + I_2$ has smaller dimension than r. Hence the leading coefficient of the Hilbert polynomial of S/I is the sum of the leading coefficients of the Hilbert polynomials of S/I_1 and S/I_2 and the result follows. $\square$

Proof.

For the third assertion, the coordinate ring of $\mathbf{P}^n$ is $S = k[x_0, \ldots, x_n]$, whose ℓth graded component has dimension $\binom{\ell+n}{n}$; the leading coefficient of this polynomial is $1/n!$. Finally, if $f \in S$ has degree d, then the ℓth graded piece of the coordinate ring of the variety defined by f has dimension $\binom{\ell+n}{n} - \binom{\ell-d+n}{n}$ for sufficiently large ℓ; the leading coefficient of this polynomial is $d/(n-1)!$ $\square$

Now we come to the main result about the intersection of a projective variety $Y \subset \mathbf{P}^n$ with a hypersurface H not containing it. We know that all components of this intersection have dimension one less than that of the variety; counting these components with the appropriate multiplicities, we get a uniform answer depending only on Y and H, and in a nice way. More precisely, let $Z_1, \ldots, Z_s$ be the components of $Y \cap H$. and let P_j be the homogeneous prime ideal of Z_i. Define the intersection multiplicity $i(Y, H; Z_j)$ of Y and H along Z_j to be the multiplicity $\mu_{P_j}(S/(I_Y + I_H))$, where S as usual denotes $k[x_1, \ldots, x_n]$ and I_y, I_H are the respective ideals of Y and H. The module $M = S/(I_Y + I_H)$ has annihilator $I_Y + I_H$, the ideal of $Y \cap H$ and P_j is a minimal prime ideal of M.

Intersection Theorem, p. 53

With notation as above, we have

$$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = \deg (Y)(\deg H).$$

Proof.

Let H be defined by the homogeneous polynomial f of degree d. We have an exact sequence of graded S-modules $0 \to (S/I_Y)(-d) \to S/I_Y \to M \to 0$, where the second map is multiplication by f. Taking Hilbert polynomials, we get $P_M(z) = P_Y(z) - P_Y(z-d)$. Now compare leading coefficients of both sides of this equation. If Y has dimension r and degree e, then $P_Y(z) = (e/r!)z^r$ plus lower order terms, whence on the right we get

$$(e/r!)z^r + \ldots - [(e/r!)(z-d)^r + \ldots] = (de/(r-1)!)z^{r-1} + \ldots \qquad \square$$

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Proof.

But now the graded module M admits a chain of submodules M_i as in the above lemma, where the quotients M_i/M_{i-1} take the form $(S/Q_i)(\ell_i)$. The Hilbert polynomial of M is then the sum of the Hilbert polynomials of the $(S/Q_i)(\ell_i)$. The degree shift by ℓ_i does not affect the leading coefficient of a Hilbert polynomial, and we can ignore that Q_i with the Hilbert polynomial of S/Q_i having degree less than $r-1$. The upshot is that the Hilbert polynomials contributing to the leading term are exactly those of the S/P_i above, each occurring $i(Y, H; Z_i)$ times with a contribution of $\deg Z_i$ each time. Comparing the two expressions for the leading coefficient, we get the desired result. $\square$

I conclude with

Corollary (Bézout's Theorem, p. 54)

Let Y, Z be distinct curves in $\mathbf{P}^2$, corresponding to homogeneous polynomials of degrees d, e. Set $Y \cap Z = \{P_1, \ldots, P_s\}$, where the P_i are points. Then we have $\sum_j i(Y, Z; P_j) = de$.

This follows because the degree of a point is easily calculated to be 1.