Lecture 5-12: Wrapping up Hensel's Lemma; analytically isomorphic singularities

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I will wrap up the proof of the Generalized Hensel's Lemma from last time. Switching then to Chapter 1 of Hartshorne's book "Algebraic Geometry", I will explore the structures of quotients of power series rings (arising from the Cohen Structure Theorem).

Proof.

Simplifying notation, set $f'(a) = e$. Then there is $h(x)$ with $f(a+ex) = f(a) + f'(a)ex + h(x))(ex)^2 = f(a) + e^2(x + x^2h(x))$. By the theorem from last time there is a ring homomorphism $\phi : R[[x]] \to R[[x]]$ that is the identity on R and sends x to $x + x^2h(x)$; by the corollary of this theorem, ϕ is an isomorphism. Applying ϕ^{-1} to the above equation, we get $f(a + e\phi^{-1}(x)) = f(a) + e^2x$. By hypothesis we have $f(a) = e^2c$ with $c \in I$. By the theorem again, there is an algebra homomorphism ψ that is the identity on R and sends x to $-c$. Applying it we get $f(x + e\psi\phi^{-1}(x)) = 0$ so $b = a + e\psi\phi^{-1}(x)$ is the desired element. $\square$

Proof.

Now suppose that e is not a zero divisor. To prove the uniqueness of b suppose that b and b_1 are roots of f differing from a by elements of eI, say $b = a + er, b_1 = a + er_1$ with $r, r_1 \in I$. By the usual theorem, there are ring homomorphisms $\beta, \beta_1 : R[[x]] \to R[[x]]$ that are the identity on R and take x to r and r_1, respectively. Applying them to the above formulas we get

$$0 = f(a + er) = f(a) + e^2(r + r^2 h(r))$$

$$0 = f(a + er_1) = f(a) + e^2(r_1 + r_1^2 h(r_1))$$

Subtracting and using the assumption that e is not a zero divisor we get $r + r^2 h(r) = r_1 + r_1^2 h(r_1)$, whence $\beta\phi(x) = \beta_1\phi(x)$. By the uniqueness statement in the theorem we get $\beta\phi = \beta_1\phi$; since ϕ is an isomorphism, $\beta = \beta_1$. Then $r = r_1$, as desired. $\square$

Turning to Chapter 1 of Hartshorne, I now look at quotients of power series rings arising from algebraic geometry. Let X be the curve in k^2 (with the field k being algebraically closed and of characteristic 0) defined by the equation $y^2 = x^2(x+1)$ and look at the origin $(0,0)$ on this curve. The coordinate ring is the quotient $k[x,y]/(y^2 - x^2(x+1))$. Completing this ring at the maximal ideal (x,y) corresponding to the point $(0,0)$ localizes it as well at this same ideal, so that the complete local ring $\hat{\mathcal{O}}_P$ of this curve at $P = (0,0)$ is the quotient $k[[x,y]]/(y^2 - x^2(x+1))$. Let's compare this curve to the pair of coordinate axes intersecting at $(0,0)$; there the corresponding complete local ring $\hat{\mathcal{O}}'_P$ is $k[[x,y]]/(xy)$. Surprisingly, these local rings are isomorphic! To prove this, I will construct a pair g,h of power series over k whose product is $p = y^2 - x^2(x+1)$. The key observation is that the lowest homogeneous component of p is $y^2 - x^2$, which is the product of distinct linear factors $y - x$ and $y + x$.

Accordingly, I form the power series $g = y + x + g_2 + g_3 + \ldots, h = y - x + h_2 + h_3 + \ldots$, where the terms g_i, h_i are power series (in fact polynomials) with lowest terms of degree i. To find g_2 and h_2, I need to have $(y - x)g_2 + (y + x)h_2 = -x^3$, which is possible since $y - x$ and $y + x$ generate the maximal ideal (x, y) of $k[[x, y]]$. To find g_3 and h_3, I need $(y - x)g_3 + (y + x)h_3 = -g_2h_2$, which is again possible, and so on; thus g and h can indeed be constructed. But now they are power series with 0 constant term and linearly independent linear terms, whence there is an automorphism of $k[[x, y]]$ sending g to x and h to y. Hence $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}'_P$ as claimed; geometrically this means that near $(0, 0)$, X looks like two lines crossing.

More generally, given two varieties X, Y of the same dimension (over the same field) and points $P \in X, Q \in Y$ we can ask whether X and Y look the same near P and Q, respectively. If the local coordinate rings $\mathcal{O}_P, \mathcal{O}_Q$ of X and Y are isomorphic, then P and Q have isomorphic neighborhoods and so the varieties X and Y are birational. Thus the local ring $\mathcal{O}_P$ carries in some sense information about almost all of X. We can get information that is "even more local" about X near P, however, by looking at the completed local ring $\hat{\mathcal{O}}_P$ (with respect to the maximal ideal corresponding to P); the additional information that this conveys is closer to what "local" means in topology or differential geometry.

Accordingly, I define P and Q to be analytically isomorphic if their completed local rings $\hat{\mathcal{O}}_P, \hat{\mathcal{O}}_Q$ are isomorphic; as above note that the completion of the coordinate ring is already local,. so one need not localize again. From the remark made last time after the Cohen Structure theorem, we see that any two smooth points on algebraic varieties of the same dimension are analytically isomorphic, so analytic isomorphism is useful primarily as a way of classifying singular points on varieties.

Generalizing the example above, let X be the curve in k^2 passing through $P = (0,0)$ defined by the equation $f(x,y) = 0$ and write $f = f_r + f_{r+1} + \ldots$, where f_i is a homogeneous polynomial in x, y of degree i. Then the lowest degree component f_r factors as a product of r linear terms; if these are distinct then we say that P is an ordinary r-fold point of X. If f_r factors as $g_s h_t$ with g_s, h_t homogeneous of respective degrees s, t and having no common factors, then it is not difficult to show that there are power series g, h beginning with g_s, h_t, respectively, whose product is f. Then it follows as above that any two ordinary double points (with $r = 2$) on any two curves are analytically isomorphic.

In fact any two ordinary triple points (with $r = 3$) on any two curves are analytically isomorphic. To see this note that by making an appropriate change of variable we may assume that the triple points correspond to the products $f_r = xy(x + ay)$ and $xy(x + by)$ with $a, b \in k^*$. Then the change of variable $x \to x, y \to (b/a)y$ converts the second product to a scalar multiple of the first one, which defines the same curve. But the pattern stops here: it is not true that any two ordinary 4-fold points are analytically isomorphic. This corresponds to a fact about the projective general linear group $G = PGL_2(k)$, which acts on the set $\mathbf{P}^1$ of lines through the origin in k^2 sharply 3-transitively. This means that, given any six such lines $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$ with $\ell \neq \ell_j, m_i \neq m_j$ for $i \neq j$, there is a unique $g \in G$ with $g\ell_i = m_i$ for all i.

What about non-ordinary points? It can be shown without too much difficulty that any double singular point on an affine curve in k^2 is analytically isomorphic to the origin for the curve $y^2 = x^r$ for a unique integer $r \geq 2$; the point is ordinary if and only if $r = 2$. If $r = 3$ the point is called a cusp; I have already discussed the curve $y^2 = x^3$ and the origin on it in some detail in previous lectures. If $r = 4$ the point is called a tacnode.

Ironically then, in effect, turning up the microscope on a curve near a singular point gives a cruder picture of the curve near the point than the one obtained from the localized coordinate ring. Nevertheless, the perspective provided by analytic isomorphisms is valuable since it tightens the analogy between varieties and differentiable manifolds: a point on such a manifold always has a neighborhood homeomorphic to a fixed space, namely $\mathbb{R}^n$.