Lecture 4-9: Primary decomposition

April 9, 2025

Today I digress to give a tool important in the study of general commutative rings (beyond polynomial rings and their quotients).

Let R be a commutative ring. There is an important generalization of the notion of prime ideal. It corresponds roughly to being a power of a prime ideal.

Definition, p. 681

A proper ideal Q of R is called *primary* if whenever $ab \in Q$ and $a \notin Q$, then $b^n \in Q$ for some positive integer n. Equivalently, all zero divisors in the quotient ring R/Q are nilpotent.

Recall that the radical $\sqrt{I}$ of an ideal I consists of all $x \in R$ with $x^n \in I$ for some n. It is an ideal containing I; I showed earlier that $\sqrt{I}$ is the intersection of all prime ideals containing I.

Now we have

Proposition 19, p. 682

1. Prime ideals are primary.
2. If Q is primary then $P = \sqrt{Q}$ is prime and is the unique smallest prime ideal containing Q (we say that Q is *P-primary* or *belongs to P*.
3. If Q is an ideal whose radical M is a maximal ideal, then Q is M-primary.
4. If M is a maximal ideal and Q satisfies $M^n \subseteq Q \subseteq M$ for some M, then Q is M-primary.

The first statement follows at once from the definition. For the second, if $ab \in \sqrt{Q}$, then $(ab)^m \in Q$ for some m, whence either $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$ by definition; since $\sqrt{Q}$ is the intersection of the prime ideals containing Q, it is the unique smallest such ideal. To prove the next statement, pass to the quotient R/Q; it suffices to show that every zero divisor in this ring is nilpotent. Thus given a ring with only one prime and thus only one maximal ideal we must show that every zero divisor is nilpotent. The intersection of the prime ideals of such ring is its nilradical, that is, the radical of the 0 ideal; given a zero divisor d in it, it is not a unit, so lies in a maximal ideal, so must be nilpotent, as desired. The last statement follows from the previous ones.

The ring-theoretic analogue of writing an algebraic variety as a finite union of irreducible varieties is then the assertion that every ideal has a *primary decomposition* in the following sense.

Definition, p. 683

An ideal I of R has a primary decomposition if it is the intersection $\cap Q_i$ of finitely many primary ideals Q_i; this decomposition is called *minimal* if no Q_i contains the intersection of the others and the Q_i have pairwise distinct radicals P_i. The P_i are called the *associated primes* of I; those P_i not containing any P_j with $j \neq i$ are also called minimal (or isolated), as are the corresponding ideals Q_i in the decomposition, The other primes P_j are called *embedded*.

The main result is then

Theorem 21, p. 684

Every ideal I in a Noetherian ring R admits a primary decomposition.

I will prove this by showing that every ideal is a finite intersection of irreducible ideals; that is, ideals I such that if $I = J \cap K$ for ideals J, K, then $I = J$ or $I = K$, and that every irreducible ideal is primary.

Proposition 20, p. 684

Every irreducible ideal in a Noetherian ring is primary. Every ideal in such a ring is a finite intersection of irreducible ones.

Proof.

First let Q be irreducible and let $ab \in Q, b \notin Q$. The set of elements $x \in R$ with $a^n x \in Q$ is easily seen to be an ideal A_n of R; since $A_1 \subseteq A_2 \subseteq \ldots$ we must have $A_n = A_{n+1} = \ldots$ for some $n > 0$. The ideals $I = (a^n) + Q$ and $J = (b) + Q$ both contain Q; if $y \in I \cap J$, then $y = a^n z + q = b + q'$ for some $q, q' \in Q$; since $ab \in Q$ we get $ay \in Q, a^{n+1} z = ay - aq \in Q, z \in A_{n+1} = A_n$. But then $a^n z, y \in Q$ and $I \cap J = Q$. Since $J \neq Q$, we must have $I = Q, a^n \in Q$, and Q is primary. Next look at the collection $\mathcal{S}$ of ideals of R that are not finite intersections of irreducible ideals; if this is not empty then it has a maximal element I, which must be reducible and thus the intersection $J \cap K$ of two properly larger ideals. Each of these must be a finite intersection of irreducible ideals, whence I is also such, a contradiction. $\square$

Since it is easy to check that a finite intersection $\cap I_i$ of P-primary ideals is again P-primary, for any prime ideal P, we may assume that the radicals of the ideals Q_i in a primary decomposition $\cap Q_i$ of I are distinct. We can then omit superfluous terms in the intersection to guarantee that no Q_i contains the intersection of the others.

Although primary decompositions are not unique, as we will soon see, there are two uniqueness results.

Theorem 21 again

Given two minimal primary decompositions $\cap_{i=1}^m Q_i = \cap_{j=1}^n Q'_j$ of the same ideal I, the sets $\{P_i\}, \{P'_i\}$ of radicals of the Q_i and Q'_i coincide.

For any $x \in R$ denote by $(I : x)$ the set of $y \in R$ with $xy \in I$. This is an ideal of R; denote its radical by $\sqrt{(I : x)}$. Then for any $x \in R$ we have $(I : x) = (\cap Q_i : x) = \cap(Q_i : x)$, whence $\sqrt{(I : x)} = \cap\sqrt{(Q_i : x)} = \cap P_i$. If $\sqrt{(I : x)}$ is prime then it must contain one of the P_i and hence coincide with it. Conversely, for each i there exists $x_i \notin Q_i, x_i \in \cap_{j \neq i} Q_j$ by minimality, whence $\sqrt{(Q_i : x)} = P_i$. Hence the P_i and P_i' are exactly the prime ideals of the form $\sqrt{(I : x)}$ for some $x \in R$. It also easily follows that the minimal prime ideals P containing I are the minimal primes P_i in any primary decomposition $\cap Q_i$ of I with $\sqrt{Q_i} = P_i$.

The other uniqueness result is

Theorem; Corollary 44, p. 717

Given two minimal primary decompositions $\cap Q_i = \cap Q'_i$ as above, the minimal primary components among the Q_i and Q'_i coincide.

The proof of this result uses localization in a crucial way. One first shows that if $S \subset R$ is multiplicatively closed and Q is P-primary, then either S intersects with P and $S^{-1}Q = S^{-1}R$ or $S \cap P = \emptyset$ and $S^{-1}Q$ is S^{-1}-primary with contraction Q to R. Then it follows that given an ideal I with primary decomposition $\cap_{i=1}^{n} Q_i$ with $\sqrt{Q_i} = P_i$ and a multiplicatively closed set S, suppose that the Q_i are numbered so that $S \cap P_i = \emptyset$ for $1 \leq i \leq t$ while $S \cap P_i \neq \emptyset$ for $i > t$; then $S^{-1}I = \cap_{i=1}^{t} S^{-1}Q_i$ is a minimal primary decomposition of $S^{-1}I$ and $S^{-1}Q_i$ is $S^{-1}P_i$-primary for $1 \leq i \leq t$. Letting S be the complement $R - P$ of a minimal prime P, then $S \cap P_i = \emptyset$ only for $P = P_i$, so the contraction of the localization of I at S is exactly the primary component Q of I belonging to P. Since the primes P_i, P_i' corresponding to the two decomposition coincide, so too do the minimal primary components.