Lecture 4-7: Localization of commutative rings

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Picking up from where I left off last time, I continue to continue to study arbitrary commutative rings rather than quotients of polynomial rings over fields, developing a basic technique that will prove quite useful for the quotients as well. In particular I use it to show that there is a finite-to-one covering map from any algebraic set V of dimension d to affine space $\mathbf{A}^d$.

Let R be a commutative ring. You know that there is a bijection between ideals of a quotient R/I and ideals of R containing I. What if one wanted to study ideals of R contained in I instead? We will see that there is indeed a way to focus attention at least on prime ideals contained in I. To do this, I generalize the construction of the field of quotients of an integral domain. Let D be a multiplicatively closed subset of R, so that (by definition) $1 \in D$ and $ab \in D$ whenever $a, b \in D$.

Definition of localization $D^{-1}R$, p. 707

The ring $D^{-1}R$, the localization of R by D, consists of all equivalence classes of ordered pairs $(d, r) \in D \times R$, subject to the relation $(d, r) \sim (e, s)$ if there is $x \in D$ with $x(er - ds) = 0$. The equivalence class of (d, r) is denoted $\frac{r}{d}$. We make $D^{-1}R$ into a ring by the usual rules for adding and multiplying fractions: $\frac{r}{d} + \frac{s}{e} = \frac{re+ds}{de}$, $\frac{r}{e}\frac{s}{d} = \frac{rs}{de}$. There is a natural map $R \to D^{-1}R$ sending r to $\frac{r}{1}$.

In constructing $D^{-1}R$ we are adjoining multiplicative inverses of the elements of D to R. If R is an integral domain and D is the nonzero elements of R, then the construction reduces to that of the field of quotients of R.

One checks immediately that the relation $\sim$ is indeed reflexive, symmetric, and transitive; note that the x appearing in the definition of $\sim$ is crucial to proving its transitivity. One also checks that the ring operations are well defined on equivalence classes. The map from R to $D^{-1}R$ is injective if and only if D contains no zero divisors; in general its kernel is the set of $r \in R$ for which there is $d \in D$ with $dr = 0$.

For our purposes the most important example of this construction occurs when D is the complement of a prime ideal P of R; then primeness of P guarantees that D is multiplicatively closed. In this case we denote $D^{-1}R$ by R_P and call it the localization of R at P. Another important example has $D = \{f^n : n \in \mathbb{N}\}$, the powers of a non-nilpotent element f of R; here $D^{-1}R$ is denoted R_f.

Ideals in a localization $D^{-1}R$ are closely related to ideals in R. More precisely, we have

Proposition, p. 709

- For any ideal of J of $D^{-1}R$ we have $J = J^{ce}$; in particular, every ideal of $D^{-1}R$ is the extension of an ideal of R and distinct ideals in $D^{-1}R$ have distinct contractions in R.
- For any ideal I of R we have $I^{ec} = \{r \in R : dr \in I \text{ for some } d \in D\}$; also $I^e = D^{-1}R$ if and only if I intersects D.
- Extension and contraction give order-preserving bijections between prime ideals of R not intersecting D and prime ideals of $D^{-1}R$.

Proof.

We already know that $J^{ce} \subseteq J$; conversely, if $\frac{a}{d} \in J$, then $a = d\frac{a}{d} \in J$, so $a \in J^c$, showing that $J^{ce} = J$. If I is an ideal of $R, r \in R$ and $d \in D$ has $dr = a \in I$ then $\frac{r}{1} = \frac{a}{d} \in I^e$, so $r \in I^{ec}$. Conversely, if $r \in I^{ec}$, then $\frac{r}{1} = \frac{a}{d}$ for some $a \in I, d \in D$, so that $x(dr - a) = 0, xdr = xa \in I$ for some $x \in D$, whence the second assertion holds. In particular, we have $I^e = D^{-1}R$ if and only if $1 \in I^{ec}$, so that I intersects D. If Q is prime in $D^{-1}R$ then we have already observed that its contraction Q^c is prime in R. Conversely, if P is prime in R and $\frac{a}{d_1}\frac{b}{d_2} \in Q = P^e$, then $\frac{ab}{d_1 d_2} = \frac{c}{d}$ for some $c \in P, d \in D$, whence $x(dab - d_1 d_2 c) = 0$ for some $x \in D$, forcing $xdab \in P$ and then $ab \in P$ since P is prime and disjoint from D. Then either $a \in P$ or $b \in P$, forcing $\frac{a}{d_1} \in Q$ or $\frac{b}{d_2} \in Q$, as desired. Since $P^{ec} = P$ we get the bijection of the third assertion. $\square$

Thus I have achieved the goal set out at the beginning: if P is prime in R, then the localization R_P is such that its prime ideals correspond bijectively to prime ideals of R lying in P. In particular, R_P has a unique maximal ideal, namely the extension $P^e = PR_P$. We call a ring R with a unique maximal ideal M local (p. 717). In this case, M consists precisely of the nonunits in R, since any nonunit x lies in a proper principal ideal (x), which can be enlarged to a maximal ideal necessarily coinciding with M.

Now I can complete the proof of Corollary 27 (p. 695), as promised last time.

Corollary 50, p. 720

Given a ring extension $R \subseteq S$ with S integral over R and P, a prime ideal of R, there is a prime ideal Q of S with $Q^c = Q \cap R = P$.

Let D be the complement of P in R. It is easy to check that $D^{-1}S$ is integral over $D^{-1}R = R_P$; let $\mathfrak{m}$ be a maximal ideal of $D^{-1}S$ (any ring has at least one maximal ideal, as we observed last term). We saw last time that the contraction $\mathfrak{m}^c = \mathfrak{m} \cap R_P$ of $\mathfrak{m}$ is maximal in R_P and thus equal to PR_P. Taking the contraction $\mathfrak{m} \cap S$ of $\mathfrak{m}$ in S, we get a prime ideal Q with $Q \cap R = P$, as desired.

Now let k be an algebraically closed field and $V \subset \mathbf{A}^n$ an algebraic set. We have seen by Noether normalization that the coordinate ring $k[V]$ is a finitely generated integral extension of $P_d = k[x_1, \ldots, x_d]$, where $d = \dim V$, whence there is a morphism π from V to affine d-space $\mathbf{A}^d$. Since points of either V or $\mathbf{A}^d$ correspond bijectively to maximal ideals of their coordinate rings, we see from the aforementioned Corollary 27 that this morphism is surjective and has finite fibers. As mentioned last time, however, It is not a topological covering map, however, since the fibers need not have the same size in general. The fibers of the map from V to $\mathbf{A}^d$ do have constant size on a Zariski open subset of $\mathbf{A}^d$, but not on the entire space.

Now let V be a variety, v a point of V. We have the maximal ideal $M = M_v$ of the coordinate ring $k[V]$ consisting of all functions vanishing at v. The localization $k[V]_M$ of $k[V]$ at M may be viewed as a subring of the function field $k(V)$; it consists of all quotients $\frac{f}{g}$ of polynomials $f, g \in k[V]$ with $g(v) \neq 0$. Such a quotient is called regular at v; the ring or all such quotients is called the local ring of V at v and is denoted $\mathcal{O}_{v,V}$ (p. 722). More generally, any quotient of polynomials $\frac{f}{g}$ defined on an open subset of V (but not necessarily on all of V) and equal in some neighborhood of a point v at which it is defined to a function in $\mathcal{O}_{v,V}$ is called regular on V; the ring of all such is denoted $\mathcal{O}_V$. The collection of regular functions defined at all points of an open subset U of V is denoted $\mathcal{O}_{U,V}$.

Example

Let V be the zero locus of $xz - yw$ in $\mathbf{A}^4$. This polynomial is easily seen to be irreducible in $k[x, y, z, w]$; since the latter ring is a UFD, the principal ideal $(xy - zw)$ is prime, so that V is a variety. The function $f = \frac{x}{y}$ is defined at all points $(x, y, z, w) \in V$ with $y \neq 0$ and is regular at such points; since $\frac{x}{y} = \frac{w}{z}$ at any point in V at which both quotients are defined, we have $f = \frac{w}{z}$ at any point in V where $z \neq 0$. Thus $f \in \mathcal{O}_V$; the domain D of definition of f consists of all points $(x, y, z, w) \in V$ at which at least one of y, z is not 0. It is easy to check that there is no single quotient $\frac{p}{q}$ of polynomials which equals f at all points of D. See p. 721.

It is not difficult to see that if R is an integral domain with field of fractions K, then the intersection $\cap_M R_M$ of all localizations of R at maximal ideals M, regarded as a subring of K, is just R (Proposition 48, p. 720). Indeed, suppose that $a \in K$ lies in the intersection. Then $I_a = \{d \in R : da \in R\}$ is an ideal of R; if it is proper then it lies in a maximal ideal M. Writing $a = \frac{r}{d}$ for some $r \in R, d \notin M$ we then get $d \in I_a$, a contradiction; so $I_a = R, a \in R$, as claimed. As a consequence, the ring $\mathcal{O}_{V,V} \cong R$ (Proposition 51, p. 722).