Lecture 4-30: The flag variety and its vanishing ideal

April 30, 2025

Last time I derived defining equations for Grassmannian varieties, using essentially the same relations (as will be observed today) as the ones used in the fall to define representations of $GL_n(\mathbb{C})$. I will continue with this program, deriving defining equations for a larger variety called the flag variety and identifying the polynomial representations of $GL_n(\mathbb{C})$ constructed in the fall as subspaces of its coordinate ring.

First I reformulate the Plücker coordinates and equations in a way that does not depend on fixing a particular basis of $\mathbb{C}^n$. Given any basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$, you saw in the fall that the dth exterior power $W = \bigwedge^d \mathbb{C}^n$ has as basis the d-fold wedges $v_{i_1} \wedge \cdots \wedge v_{i_d}$ of the v_i. Given a d-dimensional subspace S of $\mathbb{C}^n$, the dth exterior power $L = \bigwedge^d S$ is a line in W. For any indices $i_1, \ldots, i_d$ between 1 and n, rather than defining the Plücker coordinate $D_{(i_1,\ldots,i_d)}$ as I did last time, I fix a nonzero $v \in L$ and take the coefficient $c_{(i_1,\ldots,i_d)}$ of $v_{i_1} \wedge \cdots \wedge v_{i_d}$ in v when it is written as a linear combination of d-fold wedges of v_i; as with the $D_{(i_1,\ldots,i_d)}$, I decree that $c_{(i_1,\ldots,i_d)} = 0$ if two i_j are equal and changes by a sign if two i_j are interchanged.

Fixing the basis $v_1, \ldots, v_n$, one sees that the $c_{(i_1,\ldots,i_d)}$ attached to any S satisfy the same Plücker relations as the $D_{(i_1,\ldots,i_d)}$ do, and that these relations again define $\mathrm{Gr}_d(n)$ as a subvariety of $\mathbf{P}^{\binom{n}{d}-1}$. We now investigate what happens when a second subspace T of dimension $e < d$ is added to the picture. If $T \subset S$, then there is a basis $v_1, \ldots, v_e$ of T which extends to a basis $v_1, \ldots, v_d$ of S, which in turn extends to a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$. For any $k \leq e$, defining the coordinates $c_{(i_1,\ldots,i_d)}$ and $c'_{(j_1,\ldots,j_e)}$ as above relative to S and T, respectively, it is trivial to check that the Plücker relation corresponding to k and all exchanges of the first k indices among the j_r with any indices among the i_s holds.

Conversely, suppose that $T \not\subset S$. Extend a basis $v_1, \ldots v_s$ of $S \cap T$ to bases $v_1, \ldots v_d$ and $v_1, \ldots, v_s, v'_{s+1}, \ldots, v'_e$ of S and T, respectively, and finally the basis $'_{s+1}, \ldots, v'_e, v_1, \ldots, v_d$ to a basis of $\mathbb{C}^n$. Taking Plücker coordinates of T and S with respect to this last basis, one checks easily that Plücker relation with $k = 1$ does not hold, since both coordinates on its left side are 1 while all products of coordinates on its right side have leftmost factor equal to 0.

For every d with $1 \leq d \leq n$ and all tuples $(i_1, \ldots, i_d)$ of indices between 1 and n, set up Plücker coordinates $c_{(i_1,\ldots,i_d)}$. Impose the Plücker relations not only on the individual $c_{(i_1,\ldots,i_d)}$ but also on $c_{(i_1,\ldots,i_d)}$ and $c_{(j_1,\ldots,j_e)}$ as above for all $e < d$. The upshot of the foregoing discussion is then that these relations define a projective variety parametrizing all chains of subspaces $V_0 = 0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ of $\mathbb{C}^n$ such that $\dim V_i = i$. Such chains constitute (by definition) the flag variety $\mathcal{F}_n$ of $\mathbb{C}^n$. The general linear group $GL_n(\mathbb{C})$ acts transitively on $\mathcal{F}_n$; the stabilizer of the standard flag whose ith subspace V_i is spanned by the first i coordinate vectors $e_1, \ldots, e_i$ is then the subgroup B of G consisting of the upper triangular matrices in it. Thus $\mathcal{F}_n$ may be identified with the homogeneous space G/B. Its dimension is $\dim G - \dim B = \binom{n}{2}$. You may have seen this homogeneous space mentioned before in a manifolds class; the differentiable structure carried by G and B gives it a differentiable structure.

Now as it happens that you have seen these relations before: they are exactly the ones used to define the Schur module M^λ for $GL_n(\mathbb{C})$ corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ in the lecture on November 22. More precisely, the columns of the Young diagram of λ all have lengths at most n; suppose for each $j \leq n$ there are d_j columns of length j. The ideal I generated by the Plücker relations is homogeneous, so that the quotient R by this ideal has a graded structure, a typical graded piece being indexed by the tuple $(d_1, \ldots, d_n)$, Then this graded piece carries a natural action of $GL_n(\mathbb{C})$ making it isomorphic to M^λ.

Moreover, you have seen that there are elements in a suitable polynomial ring over $\mathbb{C}$ (subdeterminants of certain matrices whose entries are independent variables over $\mathbb{C}$) satisfying the Plücker relations. In particular the quotient of an appropriate polynomial ring by the ideal I generated by the Plücker relations is an integral domain. Thus I is prime and equal to the vanishing ideal corresponding to $\mathcal{F}_n$ as a projective variety. This variety is irreducible. See Chapter 9 of Fulton's book Young Tableaux.

It also follows from the above analysis that the coordinate ring R of $\mathcal{F}_n$ is isomorphic as a representation of $GL_n(\mathbb{C})$ to the direct sum of all polynomial representations of this group, each occurring exactly once. This last property turns out to be an instantiation of a general version of Frobenius reciprocity, proved for finite groups in the fall. The variety $\mathcal{F}_n$ carries a natural action of $G = GL_n(\mathbb{C})$, whence G also acts on its coordinate ring R. As a representation of $G = GL_n(\mathbb{C})$, the ring R behaves like the representation induced to G from the trivial representation of B, so that every irreducible polynomial representation M^λ of G appears in R with multiplicity equal to the dimension of the subspace S of vectors in M^λ sent to scalar multiples of themselves by elements of B (it is this subspace rather than the subspace of B-fixed vectors that is relevant because $\mathcal{F}_n$ is a projective rather than an affine variety).

Now a key fact from the representation theory of Lie groups (of which G is one) that I invoked last November to show that M^λ is irreducible is that the subspace S is one-dimensional for all partitions λ; up to scalar multiple, M^λ has a unique vector of weight λ and λ is the highest weight occurring in M^λ with respect to a suitable ordering of weights. Thus it is no surprise that R is the direct sum of all the irreducible polynomial representations of G.

I close by mentioning that in my own work I have developed analogues of the representations M^λ for other classical groups, that is, groups of linear automorphisms of $V = \mathbb{C}^n$ preserving a suitable nondegenerate bilinear form $f = (\cdot, \cdot)$. The form f, a function from $V \times V$ to $\mathbb{C}$, is linear in each coordinate and is such that the only $v \in V$ with $(v, w) = 0$ for all $w \in V$ is 0. In addition f is either symmetric, meaning that $(v, w) = (w, v)$, or skew-symmetric, meaning that $(v, w) = -(w, v)$. In the latter case the dimension $n = 2m$ of V must be even. The groups in question are the orthogonal groups $O(n, \mathbb{C})$, if the form is symmetric, or the symplectic groups $\mathrm{Sp}(2m, \mathbb{C})$, if the form is skew-symmetric. In both cases the flag variety is defined to consist of all chains of subspaces $V_0 \subset \cdots V_m$ of $\mathbb{C}^{2m}$ or $\mathbb{C}^{2m+1}$ such that $\dim V_i = i$ and all V_i are isotropic with respect to the form (so that it is identically 0 when restricted to V_i). In addition to the Plücker relations one imposes quadratic relations generating the ideal of the flag variety, corresponding to this isotropic condition.

An additional twist is that partitions are not quite enough to capture all representations in the orthogonal case. There is another group $\mathrm{Pin}(n, \mathbb{C})$, a double cover of $O_n(\mathbb{C})$, that admits certain finite-dimensional representations that do *not* carry an $O_n(\mathbb{C})$ action. These are indexed by "partitions" $(\lambda_1, \ldots, \lambda_m)$ such that each λ_i is a nonnegative integer plus $1/2$. They correspond to "tableaux" whose leftmost columns consist of half-boxes rather than boxes. In all cases semistandard tableaux or "tableaux" with entries in the appropriate set $\{\pm 1, \ldots, \pm m\}$ of integers, possibly together with 0 and suitably restricted, provide a basis for the representation corresponding to the tableau shape.