Lecture 4-28: Wrapping up prime spectra and introducing Grassmannians

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I will wrap up the discussion of prime spectra with three important examples. I then introduce Grassmannians in a complex vector space; these are important examples of projective varieties

I present two examples of prime spectra, following Examples 2 and 3 on pp. 735-6. First take $R = \mathbb{Z}[x]$, the polynomial ring in one variable over $\mathbb{Z}$. Any prime ideal P of R has prime contraction to $\mathbb{Z}$, which must be either 0 or the ideal (p) generated by a prime number p. In the first case P does not meet the multiplicatively closed set $\mathbb{Z}^*$ of nonzero integers, so it is the contraction to R of a prime ideal in $\mathbb{Q}[x]$. Any such ideal is principal, either 0 or generated by an irreducible polynomial $f \in \mathbb{Z}[x]$ which is moreover *primitive* in the sense that the greatest common divisor of its coefficients is 1; recall by Gauss's Lemma that a primitive polynomial in $\mathbb{Z}[x]$ is irreducible if and only if it is irreducible in $\mathbb{Q}[x]$. The ideal (f) is not maximal.

In the second case, where P contracts to (p), P is the preimage in R of a prime ideal in $R' = \mathbb{Z}_p[x]$, a principal ideal domain; so P must take the form (p, g), where g is a monic polynomial in R whose reduction mod p is irreducible in R'. The ideal (p, g) is then maximal.

Following the picture on p. 737, we can portray Spec R by showing how it projects by contraction to Spec $\mathbb{Z}$. For example, take $f = x^4 + 1 \in R$. This polynomial is irreducible, but becomes reducible upon reduction modulo any prime p. Modulo 2, this polynomial is the fourth power of $x + 1$, so there is just one closed point in $\mathcal{Z}(f)$ lying over $(2) \in$ Spec $\mathbb{Z}$. Modulo a prime $p \equiv 1 \mod 8$, f has four distinct roots, so there are four such closed points; modulo all other primes p, there are just two such closed points.

The picture is much the same (but perhaps geometrically more satisfying) for $R = k[x, y]$, the polynomial ring in two variables over an algebraically closed field k; the point is that R can be viewed as a polynomial ring in one variable y over the PID $k[x]$. The elements of Spec R consist of the generic point (0); the principal ideal (f) generated by an irreducible polynomial f in R, of height 1; and the closed points $(x - a, y - b)$ for $(a, b) \in k^2$. The closure of an "intermediate" point like (f) consists of this point together with the closed points corresponding to the zero locus of f.

Finally, I mention that for three or more variables all hell breaks loose; for $n \geq 3$ there are prime ideals in $k[x_1, \ldots, x_n]$ requiring arbitrarily many generators.

Shifting now to number theory, I look at the prime spectrum of a familiar Dedekind domain.

Example

First let $R = \mathbb{Z}[i]$, the ring of Gaussian integers, which some of you may have seen discussed in an undergraduate course. As previously noted, R is a PID. Its prime ideals are closely related to those of $\mathbb{Z}$; this is not surprising since R is a finitely generated integral extension of $\mathbb{Z}$. First of all, any prime $p \in \mathbb{Z}$ with $p \equiv 3 \mod 4$ remains prime in R; equivalently, the ideal (p) that it generates in R is prime. Recall that such a prime p is called inert (in R). If instead $p \equiv 1 \mod 4$, then p is the product of two primes $a \pm bi$ in R; here $a^2 + b^2 = p$ and a^2, b^2 are the only pair of integer squares adding to p. In this case p splits completely in R, corresponding to two prime ideals $(a + bi), (a - bi)$. Finally, in the exceptional case $p = 2$, the ideal p is the square of the prime ideal $(1 + i) = (1 - i)$. This prime is the only one that ramifies in R.

Example

To get a picture of Spec R, start with Spec $\mathbb{Z}$, which consists of the generic point 0 together with a closed point for each prime p. Over (contracting to) $(0) \in \operatorname{Spec} \mathbb{Z}$ one of course has only $(0) \in \operatorname{Spec} R$; over $(p) \in \operatorname{Spec} \mathbb{Z}$ one has only $(p) \in \operatorname{Spec} R$ if $p \equiv 3 \mod 4$. If $p \equiv 1 \mod 4$, $p = a^2 + b^2$, then over $(p) \in \operatorname{Spec} \mathbb{Z}$ one has $(a + bi), (a - bi) \in \operatorname{Spec} R$. Finally, over (2) one has only $(1 + i)$.

Taking a break from prime ideals, I now return to varieties, this time projective ones. I am now returning to Fulton's book Young Tableaux that I used in the fall, this time following Chapter 9. For definiteness take the basefield k to be $\mathbb{C}$, although any algebraically closed field would work equally well. Recall that I have defined projective n-space $\mathbf{P}^n$ to be affine $(n+1)$-space $\mathbf{A}^{n+1} = \mathbb{C}^{n+1}$ with the origin 0 removed, subject to the equivalence relation $(a_1, \ldots, a_{n+1}) \sim k(a_1, \ldots, a_{n+1})$ if $k \in \mathbb{C}^*$. Equivalently, one may regard $\mathbf{P}^n$ as the collection of lines through 0 in $\mathbb{C}^{n+1}$. It is natural to wonder whether the set of (vector) subspaces of $\mathbb{C}^n$ of a fixed dimension d also has the structure of a variety; the case $d = 1$ shows that one should expect this to be a projective rather than an affine variety.

The answer is yes; the variety in question is called the d-Grassmannian (in $\mathbb{C}^n$) and is denoted $\mathrm{Gr}_d(n)$. To verify this I need to attach coordinates to d-dimensional subspaces of $\mathbb{C}^n$. For this purpose recall from the fall quarter that the dth exterior power $\bigwedge^d \mathbb{C}^n$ of $\mathbb{C}^n$ is a vector space of dimension $\binom{n}{d}$; if V is a d-dimensional subspace of $\mathbb{C}^n$, then $\bigwedge^d V$ is a line lying in this subspace. So one needs a way to coordinatize exterior powers. This is easily done using determinants. Specifically, given a d-dimensional subspace V, let M be a $d \times n$ matrix whose rows form a basis of V. Then there are many choices for M, given V, but if for every d-tuple $(i_1, \ldots, i_d)$ of indices with $1 \leq i_1 < \ldots < i_d \leq n$ one lets $D_{(i_1,\ldots,i_d)}$ be the determinant of the square matrix $M_{(i_1,\ldots,i_d)}$ consisting of columns $i_1, \ldots, i_d$ of M (in that order), then the $D_{(i_1,\ldots,i_d)}$ are uniquely determined by V up to an overall nonzero scalar multiple. Since the matrix M has full rank d, not all of these subdeterminants can be 0.

By labelling each subspace V with the $\binom{n}{d}$-tuple with coordinates the $D_{(i_1,\ldots,i_d)}$, regarded as a point in the projective space $\mathbf{P}^{\binom{n}{d}-1}$, I realize $\mathrm{Gr}_d(n)$ as a subvariety of this space. To work out its dimension I normalize the matrix M attached to a subspace V, as follows. From linear algebra any two choices M_1, M_2 of M for the same V can be obtained one form another by row operations; so I can normalize M by putting it in row echelon form. The generic such form, realized on an open subset of variety of subspaces, has its first d columns equal to those of the $d \times d$ identity matrix; note that if the columns take this form, then the rows of M are automatically independent (regardless of what the other entries are) and so form a basis of a d-dimensional subspace. Accordingly the dimension of $\mathrm{Gr}_d(n)$ is $d(n-d)$. In particular one recovers the dimension of $\mathbf{P}^n$ as $n-1$.

The next step is to work out the equations defining $\mathrm{Gr}_d(n)$ as a subvariety of projective space. I first extend the definition of $D_{(i_1,\ldots,i_d)}$ in a natural way, taking it to be the determinant of the matrix $M_{(i_1,\ldots,i_d)}$ whose columns are the columns $i_1, \ldots, i_d$ of M in that order, for any indices i_j between 1 and n, not necessarily increasing or even distinct. Thus $D_{(i_1,\ldots,i_d)} = 0$ if two indices i_j, i_k are the same and it changes by a sign if i_j, i_k are interchanged. Next recall Sylvester's Lemma from the lecture on November 22 in the fall (whose notes I have sent to all of you).

It states that if N and P are two $d \times d$ matrices over a commutative ring R and if k is a number between 1 and d then one has the identity

$$(\det N)(\det P) = \sum (\det N')(\det P')$$

where the sum takes place over all matrices (N', P') obtained from (N, P) by interchanging the first k columns of P with any set of k columns in N, preserving the positions of the columns. In terms of the coordinates $D_{(i_1,\ldots,i_d)}$ introduced above, this says that

$$D_{(i_1,\ldots,i_d)} D_{(j_1,\ldots,j_d)} = \sum D_{(i'_1,\ldots,i'_d)} D_{(j'_1,\ldots,j'_d)}$$

where $(i'_1, \ldots, i'_d), (j'_1, \ldots, j'_d)$ run through all tuples of indices obtained from $(i_1, \ldots, i_d), (j_1, \ldots, j_d)$ by interchanging the first k of the j_r with any k of the i_r, preserving the positions of the indices. These are called the Plücker relations, corresponding to what I called the quadratic or exchange relations in November.

For example, if $d = 2$ and $n = 4$ then there is essentially just one Plücker relation, which says that $D_{(1,2)}D_{(3,4)} = D_{(3,2)}D_{(1,4)} + D_{(1,3)}D_{(2,4)}$. In general, given any coordinates $D_{(i_1,\dots,i_d)}$ satisfying the Plücker relations and not all 0, one can produce a $d \times n$ matrix M such that $E_{(j_1,\dots,j_d)} = \det M_{(j_1,\dots,j_d)} = D_{(j_1,\dots,j_d)}$ for all indices j_r. Start with a fixed set of indices $i_1, \dots, i_d$ such that $x = D_{(i_1,\dots,i_d)} \neq 0$; multiplying by a nonzero scalar, we may assume that $x = 1$. Then define a matrix $M = (m_{jk})$ via $m_{jk} = D_{(i_1,\dots,k,\dots,i_d)}$, where k replaces i_j as the jth coordinate. Then we have $E_{(i_1,\dots,i_d)} = D_{(i_1,\dots,i_d)} = 1$ by the construction. Now assume inductively that $E_{(j_1,\dots,j_d)} = D_{(j_1,\dots,j_d)}$ whenever $\{j_1, \dots, j_d\}$ overlaps $\{i_1, \dots, i_d\}$ in more than k indices, and let $\{j_1, \dots, j_d\}$ overlap $\{i_1, \dots, i_d\}$ in exactly k indices; rearranging, we may assume that j_1 is not one of these indices.

Applying Sylvester's Lemma in the case $k = 1$ to $M_{(i_1,\ldots,i_d)}, M_{(j_1,\ldots,j_d)}$ and the inductive hypothesis, we get $E_{(j_1,\ldots,j_d)} = D_{(j_1,\ldots,j_d)}$, as desired. Thus the Plücker relations cut out the right subvariety $\mathrm{Gr}_d(n)$ of $\mathbb{P}^{\binom{n}{d}-1}$. Also distinct subspaces have distinct Plücker coordinates. To see this note first that given a matrix M of rank d, the matrix NM corresponds to the same subspace as M for any $d \times d$ nonsingular matrix N. Thus we may assume without loss of generality, given a nonsingular $d \times d$ submatrix S of M, that $S = I$, the identity matrix. Then the recipe above for the matrix M, given the Plücker coordinates of the subspace, shows that this matrix is unique.