Lecture 4-23: Dimension theory, concluded

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I will wrap up dimension theory by summarizing the three equal integers attached to every Noetherian local ring that measure its dimension, relating this dimension to the dimensions of tangent spaces to varieties at points, defining regular local rings, and finally giving an example of Noetherian ring with infinite dimension.

First of all, I summarize the main result proved last time:

Theorem

For every Noetherian local ring R with maximal ideal M, the following three integers are equal:

1. the Krull dimension $\dim R$ of R, or the maximum length d of a strictly increasing chain $P_0 \subset \cdots \subset P_d$ of prime ideals in R;
2. the degree $d(R)$ of the characteristic polynomial $\chi_I(n)$ as a polynomial in n for large n, where $\chi_M(n)$ is the length of R/M^n as a module over itself, or equivalently the sum of the dimensions of M^i/M^{i+1} over R/M for $0 \leq i \leq n-1$;
3. the minimum number $\delta(R)$ of generators of any M-primary ideal of R.

As an example, note that the associated graded ring $G(R)$ of the localized polynomial ring $R = S_M$, M the maximal ideal $(x_1, \ldots, x_d)$ of the polynomial ring $S = k[x_1, \ldots, x_d]$, may be identified with S itself (which has a natural graded structure). Its Poincaré series is $(1-t)^{-d}$, so its dimension is d. It is easy to see directly that the ideals M_M of R and M of S both require d generators, so again the dimension of R is d.

Corollary

For any Noetherian local ring R with maximal ideal M let k be the residue field R/M. Then $\dim R \leq \dim_k(M/M^2)$.

Indeed, if $x_1, \ldots, x_s \in M$ are such that their images in M/M^2 span the latter over k, then the quotient M/M' of M by the ideal M' generated by the x_i is such that $M(M/M') = M/M'$, whence $M/M' = 0, M = M'$ by Nakayama's Lemma; since M itself is M-primary, the result follows.

Corollary 2

Let R be a Noetherian ring and $x_1, \ldots, x_r \in R$. Then any minimal prime ideal P over (containing) the ideal $I = (x_1, \ldots, x_r)$ has height at most r.

In R_P the ideal I_P becomes P-primary and so the result follows from the main theorem.

Krull's principal ideal theorem

If R is a Noetherian ring and $x \in R$ is neither a zero divisor nor a unit, then every minimal prime ideal P over (x) has height 1.

Proof.

We know that P has height at most 1. If it has height 0, then it follows from Theorem 1 of the lecture on 4-11 that P consists of zero divisors. This is a contradiction, since $x \in P$. □

The algebro-geometric corollary is that all components of the intersection of a subvariety of $\mathbf{A}^n$ of dimension d and a hypersurface in $\mathbf{A}^n$ (defined by a single equation $f = 0$, so of dimension $n - 1$) not containing it have dimension $d - 1$.

Another consequence of Krull's principal ideal theorem is

Corollary

If R is a Noetherian local ring with maximal ideal M and $x \in M$ is not a zero divisor, then $\dim R/(x) = \dim R - 1$.

Proof.

Letting $d = \dim R/(x)$, we have seen that $d \leq \dim R - 1$. On the other hand, if $x_1, \ldots, x_d$ are such that their images in $R/(x)$ generate an $M/(x)$-primary ideal, then the ideal generated by x and the x_i is M-primary in R whence $d + 1 \geq \dim R$. □

If $x_1, \ldots x_d$ generate an M-primary ideal in the Noetherian local ring R with maximal ideal then one calls the x_i a system of parameters. One has then

Proposition

Let $x_1, \ldots, x_d$ be a system of parameters for R and let Q be the M-primary ideal they generate. Let $f(t_1, \ldots, t_d)$ be a homogeneous polynomial of degree s with coefficients in R and assume that $f(x_1, \ldots, x_d) \in Q^{s+1}$. Then all coefficients of f lie in M.

Proof.

We have a surjection of graded rings $\pi : (R/Q)[t_1, \ldots, t_d] \to G_Q(R)$ sending t_i to $\bar{x}_i$, where $\bar{x}_i$ is x_i mod Q^2. The hypothesis implies that $\bar{f}$, the reduction of f mod Q, lies in the kernel of π. If some coefficient of f is not a unit in R, then $\bar{f}$ is not a zero divisor, by a HW exercise (polynomials that are zero divisors have coefficients which are all zero divisors). Hence $d(G_Q(R)) \leq d((R/Q)[t_1, \ldots, t_d]/(\bar{f}) = d((R/Q)[t_1, \ldots, t_d]) - 1 = d - 1$, since $\bar{f}$ lies in the kernel of π; but $d(G_Q(R)) = d$, a contradiction. □

We also have

Corollary

If $k \subset R$ is a field mapping isomorphically onto R/M and if $x_1, \ldots, x_d$ is a system of parameters, then $x_1, \ldots, x_d$ are algebraically independent over k.

Proof.

If $f(x_1, \ldots, x_d) = 0$, where $f \in k[t_1, \ldots, t_d]$, $f \neq 0$, then we can write $f = f_s +$ terms of higher degree, where $f_s \neq 0$ is homogeneous of degree s. Applying the previous result of f_s, we deduce that f_s has all coefficients in M, so is 0, a contradiction. □

Given a variety of dimension d, if the maximal ideal of its coordinate ring at a point is generated by d elements, then this result shows that these elements are algebraically independent.

Now I can prove the analogue for arbitrary Noetherian local rings of the characterization given earlier of smooth points of varieties.

Theorem

Let R be a Noetherian local ring of dimension d with maximal ideal M and residue field $k = R/M$. Then the following are equivalent: the graded ring $G_M(R) \cong k[t_1, \ldots, t_d]$, a polynomial ring in d variables over k; $\dim_k(M/M^2) = d$; M can be generated by d elements.

It is clear that the first property implies the second. The second implies the third by Nakayama's Lemma, as above; the third implies the first by the preceding proposition.

Rings satisfying any of these properties are called regular local. The first property shows in particular that while there are many possibilities for the coordinate ring of a smooth variety of dimension d, any two possibilities are to a first approximation "the same" near a point.

I conclude by sketching an example of a Noetherian ring with infinite dimension. Start with a polynomial ring $R = k[x_1, x_2, \ldots]$ in infinitely many variables x_i over a field k. Choose a sequence m_i of positive integers such that $m_i < m_{i+1}$ and $m_{i+1} - m_i \to \infty$ as $i \to \infty$ (e.g. take $m_i = 2^i$). For each i let P_i be the prime ideal $(x_{m_i+1}, \ldots, x_{m_{i+1}})$ and let D be the complement of the union of the P_i in R. Then D is multiplicatively closed, since each P_i is prime, and each localization $M_i = D^{-1}P_i$ has height $m_{i+1} - m_i$. Since the $m_{i+1} - m_i$ get arbitrarily large, the localization $R' = D^{-1}R$ has infinite dimension. Since a polynomial p in R can only involve finitely many of the variables x_i, the only maximal ideals of R' are the extensions M_i of the P_i and a nonzero element of R' lies in only finitely many M_i. From this it is not difficult to show that R' is Noetherian, as desired.