Lecture 4-21: Dimension theory, continued

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Continuing from last time, let R be a Noetherian local ring with maximal ideal I and Q an I-primary ideal. Last time I showed that the length $\ell(R/Q^n)$ of the quotient R/Q^n as an R/Q-module is a polynomial function $\chi_Q(n)$ of n for large n; here I am regarding R as a module over itself. I now investigate what happens if Q is replaced by another I-primary ideal Q'. First I note a general fact: in any Noetherian ring, any ideal J contains a power $\sqrt{J}^m$ of its radical $\sqrt{J}$. This follows since $\sqrt{J}$ is finitely generated, say by $x_1, \ldots, x_r$; since we have $x_i^{n_i} \in J$ for some integer n_i, the mutlnomial theorem guarantees that any combination $c = \sum r_i x_i$ has $c^N \in J$ for $N = \sum n_i$.

Then I have

Proposition

The polynomial $\chi_Q(n)$ has the same degree as $\chi_I(n)$.

Indeed, since the radical of Q is I, we must have $I^m \subseteq Q \subset I$ for some m, whence $\chi_I(n) \leq \chi_Q(n) \leq \chi_I(mn)$ for all large n; letting n go to infinity, the result follows.

Denote the common degree of $\chi_Q(n)$ for all I-primary ideal Q by $d(R)$. Now we finally have an explicit measure of the dimension of any Noetherian local ring R, namely $d(R)$. For a general Noetherian R, we define $d(R)$ to be the supremum of all $d(R_I)$, where R_I runs through all the localizations of R by maximal ideals I. It is clear that each $d(R_I)$ is finite, though I will later give an example to show that $d(R)$ can be infinite.

Now I am ready to give the other two measures of the dimension of a Noetherian local ring R with maximal ideal I, both of them turning out to equal $d(R)$. First let $\delta(R)$ be the least number of generators of any I-primary ideal Q; next recall from a previous lecture that the Krull dimension $\dim R$ of R is the largest n for which there exists a strictly increasing chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals of R. I already know that $\delta(R) \geq d(R)$ by previous results; the next goal is to show that $d(R) \geq \dim R$. To prove this I need an auxiliary result about stable Q-filtrations called the Artin-Rees lemma. Let J be any ideal in a Noetherian ring R (not necessarily local) and let M be a finitely generated R-module with a filtration (M_n). As in the special case where J is primary, we say that (M_n) is a J-filtration if $JM_n \subseteq M_{n+1}$ for all n; it is a stable J-filtration if in addition $JM_n = M_{n+1}$ for large enough n.

I now define a graded ring attached to R and graded module attached to M different from $G(R)$ and $G(M)$. Set $R^* = \oplus_{n=0}^{\infty} J^n$ (taking $J^0 = R$, as before) and $M^* = \oplus_{n=0}^{\infty} M_n$. Since $J^n M_m \subset M_{m+n}. M^*$ is a graded R^*-module. Note that R^* is Noetherian: if J is generated by $x_1, \ldots, x_r$, then R^* is generated as an $J^0 = R$-algebra by the same x_i, regarded as elements of the 1-graded piece J.

Proposition

With notation as above, the J-filtration (M_n) is stable if and only if M^* is finitely generated as an R^*-module.

Indeed, each M_n is finitely generated, whence so is each $Q_n = \oplus_{i=0}^n M_i$ as an R-module. Then Q_n generates an A^*-module, namely $M_n^* = M_0 \oplus \cdots \oplus M_n \oplus JM_n \oplus \cdots \oplus J^r M_n \oplus \cdots$. This module is finitely generated since the Q_n are finitely generated as R-modules. Since R^* is Noetherian, M^* is finitely generated if and only if the chain $M_0^* \subset M_1^* \subseteq \cdots$ stops, so that $M^* = M_{n_0}^*$. This says exactly that the filtration (M_n) is stable (starting at n_0).

Artin-Rees Lemma

Let R be a Noetherian ring, J an ideal of R, and M a finitely generated R-module equipped with a stable J-filtration (M_n). Let M' be a submodule of M. Then the induced filtration $(M' \cap M_n)$ of M' is stable.

It is immediate that $(M' \cap M_n)$ is indeed a J-filtration. It then defines a graded R^*-module which is a submodule of M^* and thus finitely generated, since R^* is Noetherian. Then the lemma follows from the preceding proposition.

Now return to the previous setting, so that R is a Noetherian local ring with maximal ideal I and Q is an I-primary ideal. Let M be a finitely generated R-module and $x \in R$ a non-zero divisor in M Set $M' = M/xM$.

Proposition

In this setting we have $d(M') \leq d(M) - 1$.

Proof.

Let $N = xM$, so that $N \cong M$ as R-modules, since x is not a zero divisor in M. Let $N_n = N \cap Q^n M$. Then for every n there is an exact sequence $0 \to N/N_n \to M/Q^n M \to M'/Q^n M' \to 0$. Setting $g(n) = \ell(N/N_n)$, we get $g(n) - \chi_Q^M(n) + \chi_Q^{M'}(n) = 0$ for all large n. By Artin-Rees, (N_n) is a stable Q-filtration of N; since $N \cong M$, $g(n)$ and $\chi_Q^M(n)$ have the same degree and leading coefficient; whence the result follows. □

The crucial result is then

Proposition

$d(R) \geq \dim R$.

Proof.

By induction on $d = d(R)$. If $d = 0$ then the length $\ell(R/I^n)$ is constant for all large n, whence $I^n = I^{n+1}$ for some n and $I^n = 0$ by Nakayama's Lemma, proved last term. Thus R is Artinian and $\dim R = 0$, by another result last term. Now suppose that $d > 0$ and let $P_0 \subset P_1 \subset \cdots \subset P_r$ be a strictly increasing chain of prime ideals in R. Let $x \in P_1, x \notin P_0$, let $R' = R/P_0$, and let x' be the image of x in R'. Then $x' \neq 0$ and R' is an integral domain, whence by the proposition we have $d(R'/(x')) \leq d(R') - 1$. Also if I' is the maximal ideal of R' then $R'/(I')^n$ is a homomorphic image of R/I^n for all n and thus has length at most that of the latter. Then $d(R) \geq d(R')$. Hence $d(R'/(x')) \leq d(R) - 1 = d - 1$. By inductive hypothesis the length of any strict chain of prime ideals in $R'/(x')$ is at most $d - 1$. But the images of the P_i for $i \geq 1$ form such a chain, whence $r - 1 \leq d - 1, r \leq d$. The result follows. □

As a beautiful and unexpected corollary we see that every Noetherian local ring has finite Krull dimension, as I implicitly indicated above. In particular, the height of any prime ideal P of R, defined to be the supremum of the lengths r of all strict chains of prime ideals $P_0 \subset \cdots \subset P_r = P$ ending at P, is finite; this is also equal to the Krull dimension of the local ring R_P. I deduce that the set of prime ideals in a Noetherian ring satisfies the descending chain condition.

I conclude by bringing minimal sets of generators into the picture. Let R be a Noetherian local ring of dimension d with maximal ideal I.

Theorem

There is an I-primary ideal of R generated by d elements $x_1, \dots, x_d$, so that $\dim R \geq \delta(R)$.

Proof.

I construct $x_1, \ldots, x_d$ inductively so that any prime ideal containing $x_1, \ldots, x_i$ has height at least i, for all $i \leq d$, as follows. Suppose that $i > 0$ and $x_1, \ldots, x_{i-1}$ have been constructed. Let $P_1, \ldots, P_s$ be the minimal prime ideals (if any) containing $x_1, \ldots, x_{i-1}$ of height exactly $i - 1$. Since $i - 1 < d = \dim R$, which is the height of I, no P_j equals I. I claim that I is not contained in the union of the P_i. Indeed, if an ideal J is contained in a finite union $\cup_{i=1}^{n} Q_i$ of prime ideals Q_i, then $J \subseteq Q_i$ for some i. This is proved by induction on n, the case $n = 1$ being clear. If it holds for $n - 1$ and $J \not\subseteq Q_i$ for all i, then for each i choose $x_i \in J, x_i \notin Q_j$ for $j \neq i$, If $x_i \notin Q_i$ for any i, then we are done; otherwise, the element $y = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} x_j$ lies in J but not in any Q_i, as desired. □

Proof.

Now let $x_i \in I, x_i \notin \cup P_i$ and let Q be any prime ideal containing $(x_1, \ldots, x - i)$. Then Q contains some minimal prime P over $(x_1, \ldots, x_{i-1})$. If $P = P_j$ for some J, then $x_i \in Q, x_i \notin P$, whence $Q \supset P$ and the height of Q is at least i; if $P \neq P_j$ for all j, then the height of P is at least i, whence again the height of Q is at least i. Thus every prime ideal containing $(x_1, \ldots, x_i)$ has height at least i, as desired. Taking $i = d$, any prime ideal P containing $(x_1, \ldots x_d)$ has height at least d, whence it must equal I, the only prime ideal of this height. Hence $J = (x_1, \ldots, x_d)$ is I-primary, as desired. □