Lecture 4-18: Dimension of Noetherian local rings

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Today I digress for a second time, interrupting the algebraic geometry that I have been doing to define the dimension of a general Noetherian ring in three ways and show that they are equivalent. I will follow the exposition in Chapter 11 of Atiyah and MacDonald's book introduction to Commutative Algebra (Wiley, 1969).

First let $R = \oplus_{n=0}^{\infty} R_n$ be a graded Noetherian ring. I will assume that the subring R_0 is also Noetherian and R is generated as an R_0-algebra by finitely many elements $x_1, \ldots, x_s$, which I take to be homogeneous, so that $x_i \in R_{k_i}$ for some positive integer k_i. (These properties actually follow automatically from the assumption on R). Let $M = \oplus_{n=0}^{\infty} M_n$ be a finitely generated graded R-module, so that $R_n M_m \subset M_{n+m}$ for all m, n and M is generated by finitely many homogeneous elements, say $m_1, \ldots, m_t$, with $m_i \in M_{r_i}$. Then M_n is finitely generated as an R_0-module, by the monomials $g_j(x) m_j$, where $g_j(x)$ is a monomial in the x_j of total degree $n - r_j$.

Let λ be an additive (integer-valued) function on the class of finitely generated R_0-modules, so that (by definition) $\lambda(B) = \lambda(A) + \lambda(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence of R_0-modules. (In the applications to algebraic geometry, R_0 will usually be a field k and $\lambda(A)$ will simply be the dimension of A over k. In what follows R_0 will turn out more generally to be an Artinian ring and $\lambda(A)$ will be the length of A as an R_0-module, defined below). Thus $\lambda(M_n)$ is defined for all M_n as above. A handy way to keep track of all the $\lambda(M_n)$ at once is to form the Poincaré series $P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n)t^n$; this is a power series with coefficients in $\mathbb{Z}$, regarded as a function of t. Then we have

Theorem (Hilbert-Serre)

The function $P(M,t)$ is a rational function of t of the form $\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$, where $f(t) \in \mathbb{Z}[t]$.

Proof.

By induction on s, the number of generators of R over R_0. In the base case $s = 0$ we have $R_n = 0$ for all large $n > 0$, whence $M_n = 0$ for all large enough n. Here $P(M, t) \in \mathbb{Z}[t]$, so the theorem follows. Now take $s > 0$ and suppose that the theorem holds for $s - 1$. Multiplication by the last generator x_s is an R_0-module homomorphism from M_n to M_{n+k_s} for all n, so one gets an exact sequence

$$0 \to K_n \to M_n \to M_{n+k_s} \to L_{n+k_s} \to 0$$

where the third map is multiplication by x_s. □

Proof.

Then $K = \oplus K_n, L = \oplus L_n$ are both finitely generated R-modules, being subquotients of M, and both are sent to 0 by x_s, so both are $R_0[x_1, \ldots, x_{s-1}]$-modules. Applying λ and breaking the exact sequence above into short exact sequences we get $\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$; multiplying by t^{n+k_s} and summing with respect to t we get $(1 - t^{k_s})P(M, t) = P(L, t) - t^{k_s}P(K, t) + g(t)$ for some polynomial $g(t)$ of degree at most $k_s - 1$. Applying the inductive hypothesis the result follows. $\square$

As a rational function of t, $P(M,t)$ has a pole at $t = 1$; the order of this pole (possibly 0), is the smallest k such that $(1-t)^k P(M,t)$ is defined at $t = 1$. It is denoted $d = d(M)$ and provides the first of our three notions of dimension. A special case is

Corollary

If all $k_i = 1$ then for all sufficiently large n, the *Hilbert function* $\lambda(M_n)$ is a polynomial in n with rational coefficients of degree $d-1$ (taking the degree of the 0 polynomial to be -1). If $d \geq 1$, the Hilbert polynomial $\lambda(M_n)$ has coefficients that are integers divided by $(d-1)!$

We have $\lambda(M_n) =$ coefficient of t^n in $f(t)(1-t)^{-s}$. Cancelling powers of $1-t$ we may assume that $s = d$ and $f(1) \neq 0$. Setting $f(t) = \sum_{k=0}^{N} a_k t^k$, since $(1-t)^{-d} = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k$, we get

$$\lambda(M_n) = \sum_{k=0}^{N} a_k \binom{d+n+k-1}{d-1}$$

for all $n \geq N$ and the sum on the right is a polynomial in n with leading term $(\sum a_k) n^{d-1}/(d-1)!$ as desired (we may assume $d > 0$). The assertion about the coefficients follows from the sum.

Returning to the exact sequence above, replace x_s by any $x \in R_k$ which is not a zero divisor in M. Then the module K in the proof of the theorem is 0 and we get $d(L) = d(M) - 1$. Thus if $x \in R_k$ is not a zero divisor in M then $d(M/xM) = d(M) - 1$. Now specialize down to the case where R_0 is an Artinian ring and $\lambda(M)$ is the length $\ell(M)$ of the finitely generated module M (that is, the integer n in any maximal strictly increasing chain $M_0 = 0 \subset \cdots \subset M_n = M$ of submodules of M; it is easy to check that this n is always finite). As an example, let R_0 be any Artinian ring and take $R = R_0[x_1, \ldots, x_s]$ to be a polynomial ring in $x_1, \ldots, x_s$ over R_0. Then R_n, the nth graded piece of R, is free over R_0 with basis consisting of the monomials $x_1^{m_1} \cdots x_s^{m_s}$ with $\sum m_i = n$. There are $\binom{s+n-1}{s-1}$ such monomials; accordingly $P(R, t) = \lambda(1-t)^{-s}$, where λ is the length of R_0 as a module over itself.

Now let R be any Noetherian local ring with maximal ideal I. Let Q be an I-primary ideal. In order to apply the preceding results, I need to replace R by a graded ring. I take this ring to be $G(R) = G_Q(R) = \oplus_{n=0}^{\infty} G_n(R) = \oplus_{n=0}^{\infty} Q^n/Q^{n+1}$. Multiplication is defined as follows: if $\bar{x}_m, \bar{x}_n$ lie in $Q^m/Q^{m+1}, Q^n/Q^{n+1}$, respectively, then $\bar{x}_m\bar{x}_n$ is taken to be the coset of $x_m x_n$ in Q^{m+n}/Q^{m+n+1}, for any representatives x_m, x_n of $\bar{x}_m, \bar{x}_n$, respectively. One readily checks that the definition does not depend on the choice of representatives. If Q is generated by $x_1, \ldots, x_s$, then $G(R)$ is generated over $G_0(R) = R/Q$ by the images of the x_i in Q/Q^2, whence $G(R)$ is Noetherian by the Hilbert basis theorem. Next let M be a finitely generated R-module and take (M_n) to be a stable Q-filtration of M. This means that $M_0 = M$, the M_n are submodules of M with $M_n \supseteq M_{n+1}$, and $QM_n \subseteq M_{n+1}$ for all n, with equality for all n sufficiently large.

I can then form $G(M) = \oplus_{n=0}^{\infty} G_n(M) = \oplus_{n=0}^{\infty} M_n/M_{n+1}$; this is a graded $G(R)$-module in a natural way. It is also Noetherian as a module: since there must be some n_0 with $M_{n_0+r} = Q^r M_{n_0}$ for all $r \geq 0$, $G(M)$ is generated by $\sum_{n \leq n_0} G_n(M)$ and each $G_n(M)$ is a finitely generated module over $G_0(R) = R/Q$; moreover, each $G_n(M)$ is of finite length λ_n over the Artinian local ring R/Q. I then have

Proposition

Let R be a Noetherian local ring with maximal ideal I and Q an I-primary ideal. Let M be a finitely generated R-module and (M_n) a stable Q-filtration of M. Then

1. M/M_n has finite length ℓ_n for all $n \geq 0$.
2. For all sufficiently large n this length ℓ_n is a polynomial $g(n)$ of degree at most s, where s is the least number of generators of Q.
3. The degree and leading coefficient of $g(n)$ depend only on M and Q, not on the filtration (M_n).

Proof.

Define $G(R), G(M)$ as above, so that $G_0(R) = R/Q$ is an Artinian local ring. As noted above, each $G_m(M) = M_m/M_{m+1}$ has finite length λ_m, whence $\ell_n = \sum_{m \le n-1} \lambda_m$ is also finite. If $x_1, \ldots, x_s$ generate Q, then then their images in Q/Q^2 generate $G(R)$ over R/Q with each image of degree 1. Hence by the corollary one has $\lambda_n = f(n)$ is a polynomial in n of degree at most $s-1$ for all large n. Since $\ell_{n+1} - \ell_n$ is a polynomial of degree at most $s-1$ for all large n, it follows that ℓ_n is a polynomial of degree at most s for all such n. Given another stable Q-filtration $(\tilde{M}_n)$ of M with corresponding polynomial $\tilde{g}(n)$, it is easy to check that there is n_0 with $M_{n+n_0} \subset \tilde{M}_n, \tilde{M}_{n+n_0} \subset M_n$ for all n, so that the ratio $g(n)/\tilde{g}(n)$ approaches 1 as $n \to \infty$. Then $g, \tilde{g}$ have the same degree and leading coefficient. $\square$

The polynomial $g(n)$ corresponding to the stable filtration $(Q^n M)$ of M is denoted $\chi_Q^M(n)$, so that $\chi_Q^M(n)$ is the length $\ell(M/Q^n M)$ of $M/Q^n M$ for all large n. If $M = R$ we write $\chi_Q(n)$ for $\chi_Q^R(n)$. This is a polynomial of degree at most s, the least number of generators of Q. Next time I will show that the degree (but not the leading coefficient) of this polynomial is independent of the choice of I-primary ideal Q.