Lecture 4-14: Krull dimension and tangent spaces

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Setting aside primary decompositions for the moment, I return to the commutative algebra leading directly to algebraic geometry. I have defined the dimension of a variety to be the transcendence degree of its function field. I also in effect gave another definition, valid for any algebraic set V, namely the maximum number of algebraically independent elements in the coordinate ring $k[V]$. I now give a more general definition of dimension for an arbitrary Noetherian ring, agreeing with this one for coordinate rings, using prime ideals rather than field theory. It will enable me to show among other things that a proper subvariety of a variety always has smaller dimension. I will then study tangent spaces of algebraic sets.

Let R be a ring (as always commutative).

Definition, p. 750

The *Krull dimension* of R, denoted $\dim R$, is the largest integer n such that there is a chain $P_0 \subset P_1 \cdots \subset P_n$ of distinct prime ideals of R; if such chains exist for arbitrarily large n, then we say that R has infinite dimension and write $\dim R = \infty$.

In the special case where $R = k[V]$ is the coordinate ring of a variety $V \subseteq \mathbf{A}^m$, any chain $P_0 \subset \cdots \subset P_n$ as in this definition corresponds via the Nullstellensatz to a chain $V_0 = V \supset \cdots \supset V_n$ of distinct subvarieties of V in $\mathbf{A}^m$.

Lemma; cf. Exercise 17, p. 704

If S is an integral extension of R then the Krull dimensions of R and S coincide.

Proof.

Given any chain $P_0 \subset \cdots \subset P_n$ of distinct prime ideals in R, we can lift it to a chain $Q_0 \subset \cdots \subset Q_n$ of prime ideals in S with $Q_i \cap P = P_i$ by Corollary 50 on p. 720 (proved last time); the Q_i are then distinct because the P_i are. There cannot be distinct prime ideals $Q \subset Q'$ of S with the same contraction P in R, for if so (by passing to R/P), we would have a nonzero prime ideal Q in an integral extension S' of an integral domain R' contracting to 0; but then if $x \in Q, x \neq 0$, satisfies the equation $x^n + \sum_{i=0}^{n-1} r_i x^i = 0$ with the r_i in R', then we cancel a power of x to get $r_0 \neq 0$, forcing $r_0 \in Q \cap R'$, a contradiction. Hence $\dim R = \dim S$, as claimed. □

The same proof also shows that if the ring R is such that all *saturated* chains $P_0 \subset \cdots \subset P_n$ of prime ideals, that is, all strictly increasing chains of primes such that no distinct prime ideals can be inserted between two consecutive terms, or at the beginning or end, have length n, then the same property holds for any integral extension S of R that is an integral domain with the same n, equal to the common Krull dimension of R and S. Next I will show that polynomial rings over fields satisfy this property.

More generally, one has

Theorem

Let $V \subset \mathbf{A}^n$ be a variety of dimension d. Then any saturated chain $P_0 \subset \cdots \subset P_m$ of prime ideals in $k[V]$ has length $m = d$.

Proof.

By induction on d. If $d = 0$, then V must be a single point and the result is obvious. In general, by Noether normalization, the ring $k[V]$ is a finite integral extension of a polynomial ring $P_d = k[y_1, \ldots, y_d]$; since I have previously observed that if all saturated chains of prime ideals in P_d have the same length, then the same is true of $k[V]$, with the same length. So I am reduced to the case where $k[V] \cong P_d$. By repeated applications of the definition of primeness, any minimal nonzero prime ideal of $k[V]$ contains an irreducible polynomial f, and then it must in fact be the principal ideal (f), which is indeed prime by unique factorization. Enlarging the singleton set $\{f\}$ to a transcendence base $\{f_1 = f, \ldots, f_d\}$ of $k(V)$, I note that $\{f_2, \ldots, f_d\}$ is a transcendence base of the quotient field of $k[V]/(f)$. An appeal to the induction hypothesis then completes the proof. □

In particular, applying the Nullstellensatz bijection, I get that any algebraic set V of $\mathbf{k}^m$ of dimension d admits a chain of subvarieties $V_0 \subset \cdots \subset V_d$ with $\dim V_i = i$. Also, any proper subvariety of a variety V of dimension d has strictly smaller dimension, since moding out by a nonzero prime ideal always lowers the transcendence degree of the field of functions. On the other hand, it is *not* true in general that any two saturated chains of prime ideals in $k[V]$ have the same length if V is not a variety (think of an algebraic set with two irreducible components of different dimensions), Using more advanced techniques, it can be shown that while $\dim R$ need not be finite for an arbitrary Noetherian ring R one has $\dim R < \infty$ if R is Noetherian local. I will prove this later. Equivalently, for any fixed prime ideal P of R, there is an upper bound on the length of any chain of distinct prime ideals ending at P. The least such upper bound is called the height of P.

In particular subvarieties of $\mathbf{A}^n$ of dimension $n-1$ have their defining ideals generated by single irreducible polynomials; such varieties are called hypersurfaces. More generally, the zero locus of any single polynomial $f \in P_n$ is such that all of its irreducible components have dimension $n-1$; there is one such component for every distinct irreducible factor of f.

Varieties have many features in common with smooth manifolds; in fact, the same word "variété" is used in French to mean both an algebraic variety and a differentiable manifold (with the adjective "differentiable" sometimes added to it for clarity in the latter case). In particular, varieties, like manifolds, have tangent spaces at each of their points. Unlike manifolds, however the dimension of the tangent space to a variety at a point need not match the dimension of the variety; at some bad points the tangent space has higher dimension. To see how this works, start with a construction quite reminiscent of one in Math 126. Recall first that partial differentiation with respect to any variable x_i is defined in any polynomial ring P_n and satisfies the usual sum and product rules for differentiation. Given $f \in P_n, a \in \mathbf{A}^n$ set $D_v f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) x_i$, a linear polynomial (see p. 723).

Definition, p. 724

Given an algebraic set A with ideal I and a point $v \in V$, the *tangent space* T_aA *to* A *at* a is the vector space $\mathcal{Z}(I')$ where I' is the ideal generated by D_af as f runs through I.

The product rule shows at once that I' is already generated by the D_af as f runs through a set $f_1, \ldots, f_m$ of generators of I. Defining the Jacobian matrix $J(av)$ of the f_i with respect to the variables $x_1, \ldots, x_n$, so that the ijth entry of J is $\frac{\partial f_i}{\partial x_j}(a)$, we can identify T_aA as the kernel of $J(a)$.

Given a choice of r rows and r columns of $J(a)$, the determinant of the corresponding minor matrix of $J(a)$ is a polynomial function, which is either identically 0 or not. Choosing r as large as possible so that some choice of r rows and columns makes this polynomial not identically zero, we see that the rank of $J(a)$ is r on an open subset N of A and strictly less than r on the complement S of N in A. The points in U are called nonsingular; those in S are called singular. Thus varieties differ from manifolds in that they can have singular points. In Math 126 we would say of any such point p that the tangent space at p is undefined; now it is always defined but sometimes has dimension higher than expected. As an example, consider again the curve C in $\mathbf{A}^2$ defined by the equation $f = x^3 - y^2 = 0$. The gradient ∇f of f vanishes only at the point $(0,0) \in C$; so this is the unique singular point of C. In particular, C is not isomorphic to the affine line $\mathbf{A}^1$.

Proposition, p. 724

For any algebraic set A and any $a \in A$, there is a natural isomorphism from the dual space $(T_a A)^*$ to the quotient $\mathfrak{M}_a/\mathfrak{M}_a^2$ of the maximal ideal $\mathfrak{m}_a \subset k[V]$ of functions vanishing at a by its square.

The function D_a has range k^n and vanishes on constant functions and on functions in M_a^2, so it induces an isomorphism from M_a/M_a^2 to a subspace of $(k^n)^*$. Restricting to $T_a A^*$ we get a surjection D from M_a to $T_a A^*$, whose kernel is easily seen to be $I + M_a^2$. The given isomorphism follows at once; I can also replace $\mathfrak{m}_a V$ here by the maximal ideal $\mathfrak{m}_{a,A}$ of the local ring $\mathcal{O}_{a,A}$ defined last time. Next time I will show that the tangent space at a generic point has dimension equal to that of the variety.