Lecture 4-11: Primary decompositions and localization of modules

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Continuing the digression, I now give some examples of primary decompositions and show how they work for modules. I then extend the technique of localization to modules.

First some examples in polynomial rings: if k is a field, then the ideal $I = (x, y)^2 \subset k[x, y]$ is primary but not irreducible, being the intersection of $(x) + I = (x, y^2)$ and $(y) + I = (x^2, y)$ (p. 684). The ideal $J = (x^2, xy)$ admits two primary decompositions, namely $(x) \cap (x, y)^2$ and $(x) \cap (x^2, y)$ (p. 685). The associated primes are (x) and $\sqrt{(x, y)^2} = (x, y)$; the only isolated prime is (x), so that a prime ideal P of $k[x, y]$ contains I if and only if it contains (x). The (x)-primary component of I corresponding to this ideal occurs in both primary decompositions; the (x, y)-primary component, by contrast, differs between the two decompositions. The associated variety of (x) is the y-axis; it contains that of (x, y), which is just the origin. In general, the associated varieties of the embedded primes disappear in some sense, being contained in the associated varieties of the isolated primes.

Now let M be a module over a general commutative ring R. An element x of R is said to be a zero divisor in M if scalar multiplication by x is not injective on M; similarly x is said to be nilpotent in M if multiplication by x acts nilpotently on M. We say that M is primary if every zero-divisor in M is nilpotent; more generally, if N is a submodule of M, then N is primary in M if M/N is primary. If N is primary in M then $(N : M) := \{x \in R : xM \subseteq N\} = \text{Ann}\, M/N$ is a primary ideal of R, so that its radical $P = \sqrt{(N : M)}$ is prime; we say that N is P-primary in M in this case.

A primary decomposition of N in M is then a finite intersection $\cap N_i$ of primary submodules of M equalling N. It is called *minimal* if the radicals $P_i = \sqrt{(N_i : M)}$ are distinct and no N_i contains the intersection of the others. In this case the primes P_i are said to belong to N in M. Once again the P_i not containing any other P_j are called *isolated* while the other P_i are called *embedded*.

Now if R is Noetherian and M is finitely generated over R, then M is Noetherian as a module, so that it satisfies the ascending chain condition on submodules, or equivalently every submodule of M is finitely generated. To prove this it suffices to show that the free module R^n is Noetherian; this follows easily by induction on n and the short exact sequence $0 \to R \to R^n \to R^{n-1} \to 0$. Then we have

Primary Decomposition Theorem for modules

Any submodule of a finitely generated module M over a Noetherian ring R has a primary decomposition.

This is proved in the same way as the corresponding result for ideals. One first defines the notion of irreducible submodule of M as one not realizable as the intersection of two submodules properly containing it (NOT the same as our previous definition of irreducible module, which we will have no further occasion to use). Then every submodule is a finite intersection of irreducible submodules and every irreducible submodule is primary, so the result follows.

As before the set of minimal prime ideals belonging to a fixed submodule N of M with a primary decomposition in M (those not containing other such prime ideals) is uniquely determined by M and N. These prime ideals may be described directly: they are the associated primes of M, that is, annihilators in R of some element of M that are prime ideals (see p. 670, in the exercises after Section 15.1). Denote by Ass M the set of associated primes of M. We then have

Theorem 1

Let M be finitely generated over a Noetherian ring R. Then Ass M is finite and consists of primes containing the annihilator Ann M of M. It includes all primes minimal among primes containing Ann M. Moreover, the union of the associated primes of M consists of 0 and the set of zero-divisors of M.

To prove the first assertion I first prove an auxiliary result given below, of interest in its own right. I then extend localization of rings, discussed in the lecture on April 7, to modules, and use that to prove the rest of the theorem. Throughout M will be a finitely generated module over a Noetherian ring R.

Theorem 2

There is a finite chain of submodules $M_0 = 0 \subset M_1 \subset \cdots \subset M_n = M$ such that each quotient $M_i/M_{i-1} \cong R/P_i$ for some prime ideal P_i.

Proof.

I first show that M has a submodule isomorphic to R/P_1 for some prime P_1. Choose $m \in M$ whose annihilator $P = \text{Ann}\, m$ is maximal among annihilators of nonzero elements of M. If $ab \in P$, $a \notin P$, then Ann am, which contains P, must equal it by maximality, forcing $bm = 0$ and $b \in P$, so $P_1 = P$ is indeed prime. Letting $M_1 = Rm, M' = M/M_1$, choose nonzero $m_2 \in M'$ whose annihilator P_2 is maximal among annihilators of elements and thus prime. Letting M_2 be the preimage of Rm_2 in M, continue in this way to produce a chain of submodules $M_1 \subset M_2 \subset \cdots$. The chain terminates after finitely many steps with $M_n = M$, since M is Noetherian, and has the desired property. □

It is easy to check that given a short exact sequence $0 \to M' \to M \to M" \to 0$ of R-modules we have Ass $M \subset$ Ass $M' \cup$ Ass $M"$. Given a prime ideal P, the annihilator of any nonzero $x \in R/P$ is P; so the first assertion of Theorem 1 follows (it is clear that any $P \in$Ass M contains Ann M). Given a zero divisor x, lying in say Ann m for nonzero $m \in M$, replace m as above by a nonzero multiple of itself with maximal annihilator among annihilators of such multiples. Then P is prime, as we saw above, and indeed an associated prime of M; conversely any associated prime clearly consists of zero divisors.

As promised above, I now extend localization to modules. Let D be a multiplicatively closed subset of a general commutative ring R and M an R-module.

Definition

The localization $D^{-1}M$ consists of equivalence classes of ordered pairs $(d, m) \in D \times M$ subject to the relation $(d, m) \sim (e, n)$ if there is $d' \in D$ with $d'(em - dn) = 0$. Two such pairs $(d, m), (e, n)$ are added and subtracted as for $D^{-1}R$. We make $D^{-1}M$ into a $D^{-1}R$ module via the recipe $(d, r)(e, m) = (de, rm)$. If P is a prime ideal of R, then M_P, the *localization of M at P*, denotes $D^{-1}M$ with $D = R - P$. It is a module over R_P.

For an ideal I in a ring R and a multiplicatively closed subset D, we have $D^{-1}I = D^{-1}R$ if and only if D intersects I. The corresponding fact for modules is that given a submodule N of a finitely generated module M, we have $D^{-1}N = D^{-1}M$ if and only if some $d \in D$ lies in $(N : M)$. Indeed, it is clear that this condition is sufficient; conversely, if $D^{-1}N = D^{-1}M$ and $m_1, \ldots, m_n$ generate M, then there are $d_1, \ldots, d_n \in D$ with $d_i m_i \in N$ and $d_1 \ldots d_n M \subset N$. Then it is not difficult to check that primary decompositions of modules behave well under localization, so that if N is a submodule of an R-module M admitting a decomposition $\cap N_i$ with N_i being P_i-primary in R and if D is multiplicatively closed in R, then a primary decomposition of $D^{-1}N$ in $D^{-1}M$ is given by $\cap D^{-1}N_i$, the intersection taking place over the N_i with D not intersecting P_i.

Finally I prove the second assertion in Theorem 1. Note first that given any ideal I in a commutative ring R, there are prime ideals P of R minimal over I by Zorn's Lemma, since the prime ideals of R containing I are partially ordered by reverse inclusion and the intersection of a chain of prime ideals is prime. If moreover R is Noetherian, then there are only finitely many minimal primes containing a fixed ideal I: these are the associated primes of R/I. If in addition M is finitely generated over R and P is a minimal prime over Ann M, then we can localize R and M at P. The set Ass M_P is nonempty and P_P is the only prime ideal in R_P annihilating an element of M_P, so we must have $P \in$Ass M, as claimed.