HW #7

Math 506A

1. (20 points) Let R be the ring of continuous real-valued functions on the unit interval $S = [0, 1]$. Use the compactness of S to show that given any proper ideal I of R, the set $Z(I) \subseteq S$ of common zeros of functions in I is nonempty. On the other hand, show that distinct radical ideals I of R can have the same $Z(I)$.

2. Let k be a field. Show that the ideal $(x^n, x^{n-1}y, \ldots, y^n)$ in $k[x, y]$ cannot be generated by fewer than $n + 1$ elements.

3. (20 points) Classify the primes in the ring $R = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, as follows. You may assume that R is a PID and thus a UFD. Recall that the *norm* $N(a+bi)$ of a Gaussian integer $a + bi$ is the product $(a + bi)(a - bi) = a^2 + b^2$ and that $N(zw) = N(z)N(w)$ for $z, w \in R$. Using norms, show that every prime in R is either a multiple of a prime $p \in \mathbb{Z}$ by a unit (± 1 or $\pm i$) or else has prime norm. Show that a prime $p \equiv 3 \mod 4$ in $\mathbb{Z}$ remains prime in R. Show that a prime $p \equiv 1 \mod 4$ in $\mathbb{Z}$ factors in R as $(a + bi)(a - bi)$ for some a, b, by first showing that the equation $x^2 + 1 = 0$ has a solution in the modular integers $\mathbb{Z}_p$. Finally, deal with the case $p = 2$.