HW #4

Math 506A

1. Prove or disprove: if two representations π, π' of a finite group G are such that $\det \pi(g) = \det \pi'(g)$ for all $g \in G$, then π and π' are equivalent.

2. Let π be an irreducible representation of a finite group G with character χ. Show that $\chi(g) = \chi(1)$ if and only if g lies in the kernel of the representation π, and that if $g \neq 1$ then there is an irreducible character χ with $\chi(g) \neq \chi(1)$.

3. Let G be a finite group and H a subgroup such that $g^{-1}Hg \cap H = 1$ for $g \notin H$. Show that the set N of all elements in G not conjugate to any element of H, together with 1, is a normal subgroup of G, using the following steps. First show that if c_H is a class function on H with $c_H(1) = 0$, then the induced function $c_G = \mathrm{Ind}_H^G c_H$ has the same square length as c_H. Next let χ_H be an irreducible character of H of degree $d = \chi(1)$ and let c_χ be the function induced from $\chi_H - d$ to G. Use Frobenius reciprocity to show that c_χ takes the form $\chi_G - d = \chi_G - \chi_G(1)$ for some irreducible character χ_G of G. Letting χ_H run through the irreducible characters of H, show that N is the set of common zeros of the c_χ in G and thus the intersection of the kernels of the representations π_G with character χ_G.

4. (20 points) For any partition λ of n, the construction of the Specht module S^λ makes sense over any basefield k. Show that if k has characteristic 3 and $\lambda = (2,1)$, then $V = S^\lambda$ is *not* irreducible over S_3, by working out the actions of a 3-cycle and a transposition explicitly on the basis of V parametrized by standard tableaux of shape λ.