HW #3

Math 506A

1. Construct character tables for the dihedral and quaternion groups of order 8, making sure to identify and label the conjugacy classes for these groups.

2. (a) Show that if g is any element of the symmetric group S_n of order m, then any power g^a is conjugate to g in S_n whenever a and m are relatively prime.

(b) Recalling the Galois group of the polynomial $x^m - 1$ over $\mathbb{Q}$ computed last quarter, use the previous part to show that $\chi(g) \in \mathbb{Q}$ for any irreducible (complex) character χ of S_n.

(c) Deduce that in fact $\chi(g) \in \mathbb{Z}$ for any irreducible character χ of S_n.

3. Recall from last quarter the *tensor product* $U = V \otimes_{\mathbb{C}} W$ of two finite-dimensional vector spaces V, W over $\mathbb{C}$; this is a complex vector space whose dimension is the product mn of the dimensions m and n of V and W. If V, W are modules for the same finite group G, then U becomes a G-module via the recipe $g(v \otimes w) = gv \otimes gw$ for $g \in G, v \in V, w \in W$. Show that the character χ_U of U is the product $\chi_V \chi_W$ of the characters of V and W.

4. More generally, let V, W be modules for the groups G, H, respectively. Make $U = V \otimes W$ into a module for the direct product $G \times H$ via the recipe $(g, h)(v \otimes w) = gv \otimes hw$. Show that U is irreducible over $G \times H$ whenever V and W are irreducible over G and H.

5. Let $G = D_{2n}$ be the group of symmetries of a regular $2n$-gon, so that G has order $4n$. Recall that G is generated by two elements x, y such that $x^{2n} = y^2 = 1, yxy = x^{-1}$. Determine the degrees of the irreducible representations of G, as follows. Show first that $x^2 \in G$ acts trivially on any one-dimensional representation and use this to count the number of such representations. Then count conjugacy classes in G to determine the number and degrees of the remaining representations.