HW #2

Math 506A

1. Let R be any ring and M an irreducible left R-module (so that M has no nonzero proper R-submodule). Show that the ring $\hom_R(M, M)$ of R-module homomorphisms from M to itself is a division ring; that is, it satisfies all axioms of a field except for commutativity of multiplication.

2. Recall that the dihedral group D_8 of order 8 is generated by two elements x, y such that $x^4 = y^2 = 1, yxy = yxy^{-1} = x^{-1}$. This group is the group of symmetries of a square. Use this fact to choose vertices of a suitable square and write down an irreducible two-dimensional real representation π of D_8, computing the 2×2 matrices $\pi(x)$ and $\pi(y)$ explicitly.

3. Recall that the quaternion group Q_8, also of order 8 and generated by two elements x, y, this time such that $x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}$, is *not* isomorphic to any subgroup of the group $G = GL(2, \mathbb{R})$ of 2×2 invertible real matrices. (Assume contrarily that π is an isomorphism from Q_8 into a subgroup of G. Show first that $\pi(x^2) = -I$, the negative of the identity matrix. Replacing π by a conjugate, you may assume that $\pi(x)$ is in rational canonical form. Then deduce a contradiction).

4. (20 points) Let V be the infinite-dimensional module of the Klein four-group $K = C_2 \times C_2$ over the field F_2 with two elements given in class, so that V has a basis $\{v_i : i \in \mathbb{Z}\} \cup \{w_i : i \in \mathbb{Z}\}$ over F_2 and the commuting generators x, y of K are such that $xw_i = yw_i = w_i, xv_i = v_i + w_{i+1}, yv_i = v_i + w_i$. Show that V is indecomposable. (Assume for a contradiction that $V = V_1 \oplus V_2$ for some nonzero submodules V_1, V_2. Show that the projections P_1, P_2 of V_1, V_2 to the span S of the v_i intersect trivially and that S is their direct sum. Then $(x-1)P_1 + (x-1)P_2 = (y-1)P_1 + (y-1)P_2 = T$, the span of the w_i, forcing $(x-1)P_1 = (y-1)P_1, (x-1)P_2 = (y-1)P_2$. Whenever a particular combination $\sum a_i v_i$ lies in P_1, this last fact forces a large number of other such combinations to lie in P_1, and similarly for P_2. Deduce a contradiction unless one of P_1, P_2, say P_1, is all of S. Then show that $V_1 = V$.)