HW #8, due 5-30

Math 506A

1. (30 points) Let A, B be rings with $A \subset B$ and assume that B is flat as an A-module. Show that the following are equivalent : (i) $BI \cap A = I$ for all ideals I of A; (ii) the contraction map from Spec B to Spec A is surjective; (iii) for every maximal ideal M of A one has $BM \neq B$; (iv) for any nonzero A-module we have $M_B = B \otimes_A M \neq 0$; (v) for every A-module M the map from M to M_B sending x to $1 \otimes x$ is injective. (Show along the way that for any ring homomorphism $A \to B$ and any prime ideal P of A one has that P is the contraction of a prime ideal of B if and only if $BP \cap A = P$, and that for any B-module N, regarded as an A-module by extension of scalars, the map $g : N \to N_B$ mapping y to $1 \otimes y$ is injective and $g(N)$ is a direct summand of N_B.)

2. (20 points) Let A be the ring of germs of smooth real-valued functions on $\mathbb{R}$ at $x = 0$ (so that any two functions in A are identified if they agree on a neighborhood of 0). Identify the completion $\hat{A}$ of A with respect to its maximal ideal and show that $\hat{A}$ is Noetherian even though A is not. (You may use Borel's Theorem that any power series is the Taylor series of a smooth function at $x = 0$.)